Lecture 16: Minimum Spanning Trees

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601.433/633 Introduction to Algorithms

Introduction

Definition

A *spanning tree* of an undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is a set of edges $\mathbf{T} \subseteq \mathbf{E}$ such that $(\mathbf{V}, \mathbf{T})$ is connected and acyclic.

Definition

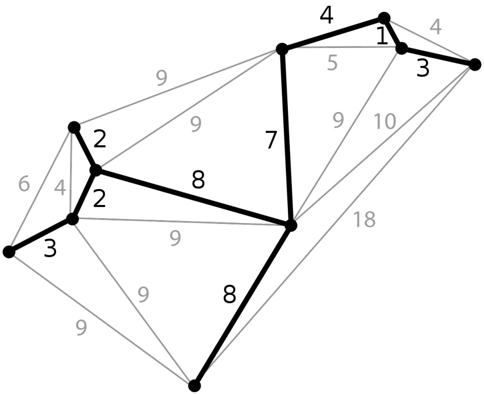
Minimum Spanning Tree problem (MST):

- Input:
  - Undirected graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$
  - Edge weights $\mathbf{w} : \mathbf{E} \to \mathbb{R}_{\geq 0}$
- Output: Spanning tree minimizing $\mathbf{w}(\mathbf{T}) = \sum_{e \in \mathbf{T}} \mathbf{w}(\mathbf{e})$.

Foundational problem in *network design*. Tons of applications.

Today: one "recipe", two different algorithms from recipe. Main idea: greedy.

Examples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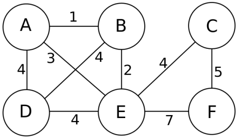
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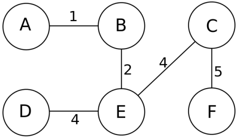
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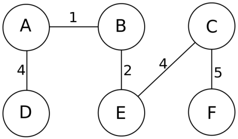
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Generic Algorithm

Generic Greedy

Definition

Suppose that $\mathbf{A}$ is subset of *some* MST. If $\mathbf{A} \cup \{\mathbf{e}\}$ is also a subset of some MST, then $\mathbf{e}$ is *safe* for $\mathbf{A}$.

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Generic-MST {
    A = ∅
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        find an edge e safe for A
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Theorem

Generic-MST is correct: it always returns an MST.

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Induction.

Claim: $\mathbf{A}$ always a subset of some MST.
Base case: ✓
Inductive step: ✓ □

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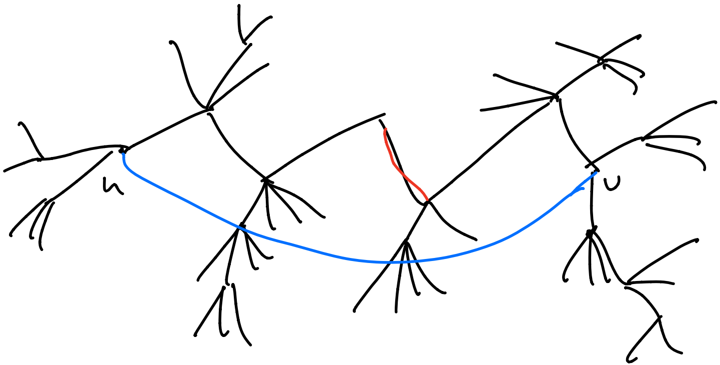
But how to find a safe edge? And which one to add?

Structural Properties

Lemma

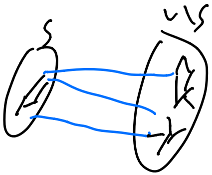
Let $\mathbf{T}$ *be a spanning tree, let* $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, *and let* $\mathbf{P}$ *be the* $\mathbf{u}-\mathbf{v}$ *path in* $\mathbf{T}$. *If* $\{\mathbf{u}, \mathbf{v}\} \notin \mathbf{T}$, *then* $\mathbf{T}' = (\mathbf{T} \cup \{\{\mathbf{u}, \mathbf{v}\}\}) \setminus \{\mathbf{e}\}$ *is a spanning tree for all* $\mathbf{e} \in \mathbf{P}$.



Structural Properties

Definition

A *cut* $(S, V \setminus S)$ (or $(S, \bar{S})$ or just S) is a partition of V into two parts. Edge e *crosses* cut $(S, \bar{S})$ if e has one endpoint in S and one endpoint in $\bar{S}$.



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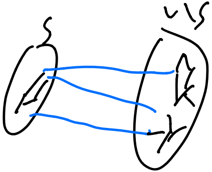
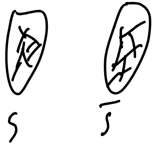
Cut $(\mathbf{S}, \bar{\mathbf{S}})$ *respects* $\mathbf{A} \subseteq \mathbf{E}$ if no edge in $\mathbf{A}$ crosses $(\mathbf{S}, \bar{\mathbf{S}})$





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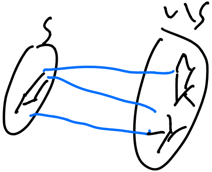

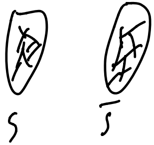

Definition

$\mathbf{e}$ is a *light edge* for $(\mathbf{S}, \bar{\mathbf{S}})$ if $\mathbf{e}$ crosses $(\mathbf{S}, \bar{\mathbf{S}})$ and $\mathbf{w}(\mathbf{e}) = \min_{\mathbf{e}' \text{ crossing } (\mathbf{S}, \bar{\mathbf{S}})} \mathbf{w}(\mathbf{e}')$

Main Structural Theorem

Theorem

Let $\mathbf{A} \subseteq \mathbf{E}$ *be a subset of some MST* $\mathbf{T}$*, let* $(\mathbf{S}, \bar{\mathbf{S}})$ *be a cut respecting* $\mathbf{A}$*, and let* $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\}$ *be a light edge for* $(\mathbf{S}, \bar{\mathbf{S}})$*. Then* $\mathbf{e}$ *is safe for* $\mathbf{A}$*.*

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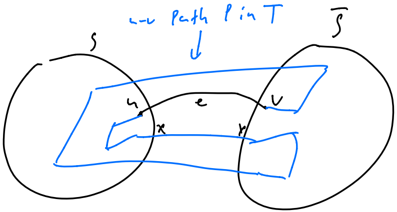
$\implies$ $\mathbf{T}'$ a spanning tree by first lemma



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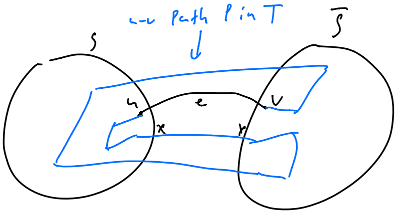

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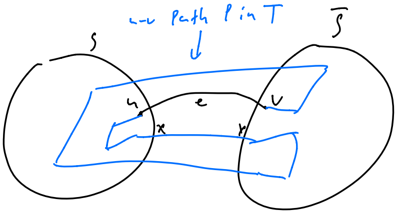

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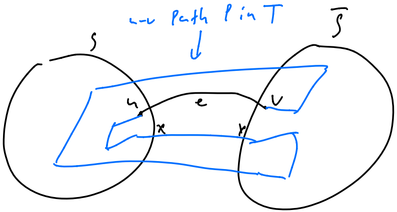

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Idea: start at arbitrary node $\mathbf{u}$. Greedily grow MST out of $\mathbf{u}$.

$\mathbf{A} = \varnothing$
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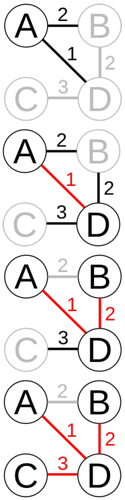

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- $(\mathbf{S}, \bar{\mathbf{S}})$ always respects $\mathbf{A}$ (induction).
- If edge $\mathbf{e}$ added then light for $(\mathbf{S}, \bar{\mathbf{S}})$
- Hence $\mathbf{e}$ safe for $\mathbf{A}$ by main structural theorem.

□

Running Time

Trivial analysis:

- Every spanning tree has $\mathbf{n-1}$ edges $\Longrightarrow$ $\mathbf{n-1}$ iterations
- In each iteration, look through all edges to find min-weight edge crossing $(\mathbf{S}, \bar{\mathbf{S}}) \Longrightarrow$ $\mathbf{O(m)}$ time
- Total $\mathbf{O(mn)}$

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- $\mathbf{n}$ Inserts, $\mathbf{n}$ Extract-Mins, $\mathbf{m}$ Decrease-Keys
- Like Dijkstra, $\mathbf{O(m \log n)}$ using binary heap. $\mathbf{O(m + n \log n)}$ with Fibonacci heap (only Extract-Min is logarithmic)

Kruskal's Algorithm

Algorithm

Intuition: can we be *even greedier* than Prim's Algorithm?

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$\mathbf{A} = \emptyset$
Sort edges by weight (small to large)
For each edge $\mathbf{e}$ in this order $\{$
    if $\mathbf{A} \cup \{\mathbf{e}\}$ has no cycles, $\mathbf{A} = \mathbf{A} \cup \{\mathbf{e}\}$
$\}$
return $\mathbf{A}$

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Theorem

Kruskal's algorithm computes an MST.

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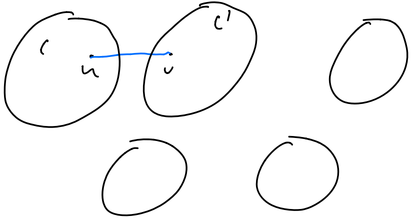

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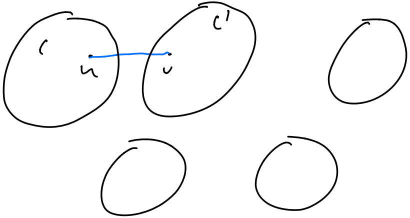

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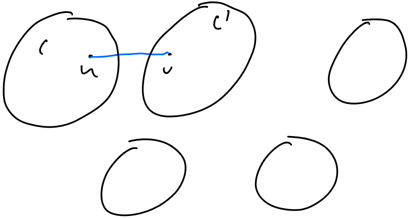

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Main structural theorem $\implies$ $\{\mathbf{u}, \mathbf{v}\}$ safe for $\mathbf{A}$

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- $O(m(m+n)) = O(m^2 + mn)$

Can we speak this up with data structures?

Running Time

Sorting edges: $\mathbf{O(m \log m) = O(m \log n)}$
Easy analysis: $\mathbf{m}$ iterations, DFS/BFS in each iteration to check if endpoints already connected.

- $\mathbf{O(m(m+n)) = O(m^2 + mn)}$

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Union-Find! Connected components of $\mathbf{A}$ are disjoint sets.

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Union-Find! Connected components of $\mathbf{A}$ are disjoint sets.

- Make-Sets:

Running Time

Sorting edges: $\mathbf{O(m \log m) = O(m \log n)}$
Easy analysis: $\mathbf{m}$ iterations, DFS/BFS in each iteration to check if endpoints already connected.

- $\mathbf{O(m(m+n)) = O(m^2+mn)}$

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Union-Find! Connected components of $\mathbf{A}$ are disjoint sets.

- Make-Sets: $\mathbf{n}$

Running Time

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Union-Find! Connected components of $\mathbf{A}$ are disjoint sets.

- Make-Sets: $\mathbf{n}$
- Finds:

Running Time

Sorting edges: $O(m \log m) = O(m \log n)$
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- $O(m(m+n)) = O(m^2 + mn)$

Can we speak this up with data structures?

Union-Find! Connected components of A are disjoint sets.

- Make-Sets: n
- Finds: $2m$

Running Time

Sorting edges: $O(m \log m) = O(m \log n)$
Easy analysis: m iterations, DFS/BFS in each iteration to check if endpoints already connected.

- $O(m(m+n)) = O(m^2 + mn)$

Can we speak this up with data structures?

Union-Find! Connected components of A are disjoint sets.

- Make-Sets: n
- Finds: $2m$
- Unions:

Running Time

Sorting edges: $\mathbf{O(m \log m) = O(m \log n)}$
Easy analysis: $\mathbf{m}$ iterations, DFS/BFS in each iteration to check if endpoints already connected.

- $\mathbf{O(m(m+n)) = O(m^2 + mn)}$

Can we speak this up with data structures?

Union-Find! Connected components of $\mathbf{A}$ are disjoint sets.

- Make-Sets: $\mathbf{n}$
- Finds: $\mathbf{2m}$
- Unions: $\mathbf{n-1}$

Running Time

Sorting edges: $O(m \log m) = O(m \log n)$
Easy analysis: m iterations, DFS/BFS in each iteration to check if endpoints already connected.

- $O(m(m+n)) = O(m^2 + mn)$

Can we speak this up with data structures?

Union-Find! Connected components of A are disjoint sets.

- Make-Sets: n
- Finds: $2m$
- Unions: $n - 1$

$O(m \log^* n)$ using union-by-rank + path compression
$O(m + n \log n)$ using list data structure

Running Time

Sorting edges: $\mathbf{O(m \log m) = O(m \log n)}$
Easy analysis: $\mathbf{m}$ iterations, DFS/BFS in each iteration to check if endpoints already connected.

- $\mathbf{O(m(m+n)) = O(m^2 + mn)}$

Can we speak this up with data structures?

Union-Find! Connected components of $\mathbf{A}$ are disjoint sets.

- Make-Sets: $\mathbf{n}$
- Finds: $\mathbf{2m}$
- Unions: $\mathbf{n-1}$

$\mathbf{O(m \log^* n)}$ using union-by-rank + path compression
$\mathbf{O(m + n \log n)}$ using list data structure

Sorting dominates! $\mathbf{O(m \log n)}$ total.