# Lecture 15: All-Pairs Shortest Paths

Michael Dinitz

October 19, 2021
601.433/633 Introduction to Algorithms

# Announcements

- HW5 due now
- HW6 due next Thursday
- Mid-Semester feedback on Campuswire!

# Introduction

Setup:

- Directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$
- Length $\ell(\mathbf{x}, \mathbf{y})$ on each edge $(\mathbf{x}, \mathbf{y}) \in \mathbf{E}$
- Length of path $\mathbf{P}$ is $\ell(\mathbf{P}) = \sum_{(\mathbf{x},\mathbf{y}) \in \mathbf{P}} \ell(\mathbf{x}, \mathbf{y})$
- $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \mathbf{min}_{\mathbf{x} \to \mathbf{y} \text{ paths } \mathbf{P}} \ell(\mathbf{P})$

Last time: All distances from source node $\mathbf{v} \in \mathbf{V}$.

Today: Distances between all pairs of nodes!

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- No negative weights: $\mathbf{n}$ runs of Dijkstra, time $\mathbf{O}(\mathbf{n}(\mathbf{m} + \mathbf{n} \log \mathbf{n}))$
- Negative weights: $\mathbf{n}$ runs of Bellman-Ford, time $\mathbf{O}(\mathbf{nmn}) = \mathbf{O}(\mathbf{mn}^2)$

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Can we do better? Particularly for negative edge weights?

- Main goal today: Negative weights as fast as possible.

# Floyd-Warshall Algorithm

# Floyd-Warshall: A Different Dynamic Programming Approach

To simplify notation, let $\mathbf{V} = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}\}$ and $\ell(\mathbf{i}, \mathbf{j}) = \infty$ if $(\mathbf{i}, \mathbf{j}) \notin \mathbf{E}$

Bellman-Ford subproblems: length of shortest path with at most some number of edges

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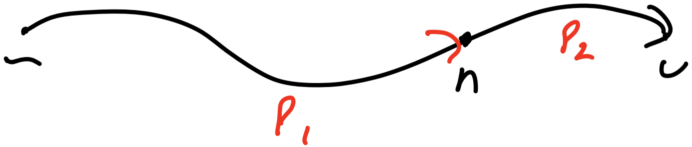
New subproblems:

- Intuition: "shortest path from **u** to **v** either goes through node **n**, or it doesn't"
  - If it doesn't: shortest uses only first nodes in $\{1, 2, \ldots, n-1\}$.
  - If it does: consists of a path $\mathbf{P_1}$ from **u** to **n** and a path $\mathbf{P_2}$ from **n** to **v**, neither of which uses **n** (internally).



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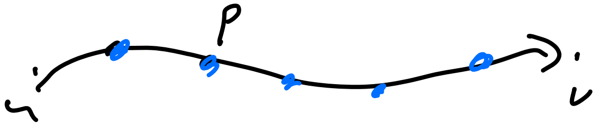
- Subproblems: shortest path from $u$ to $v$ that only uses nodes in $\{1, 2, \ldots k\}$ for all $u, v, k$.

# Formalizing Subproblems




$\mathbf{u} \to \mathbf{v}$ path $\mathbf{P}$: "intermediate nodes" are all nodes in $\mathbf{P}$ other than $\mathbf{u}, \mathbf{v}$.

$\mathbf{d_{ij}^k}$: distance from $\mathbf{i}$ to $\mathbf{j}$ using only $\mathbf{i} \to \mathbf{j}$ paths with intermediate vertices in $\{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}\}$.

- Goal: compute $\mathbf{d_{ij}^k}$ for all $\mathbf{i}, \mathbf{j}, \mathbf{k} \in [\mathbf{n}]$.
- Return $\mathbf{d_{ij}^n}$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{V}$.

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$$\mathbf{d_{ij}^k} = \begin{cases} & \text{if } \mathbf{k} = \mathbf{0} \\ & \text{if } \mathbf{k} \geq \mathbf{1} \end{cases}$$

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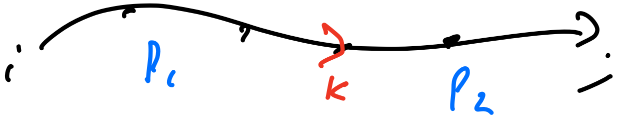


if k not used: $d_{ij}^{k-1}$

k is used: $d_{ik}^{k-1} + d_{kj}^{k-1}$

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$$d_{ij}^k = \begin{cases} \ell(i,j) & \text{if } k = 0 \\ \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1}) & \text{if } k \geq 1 \end{cases}$$

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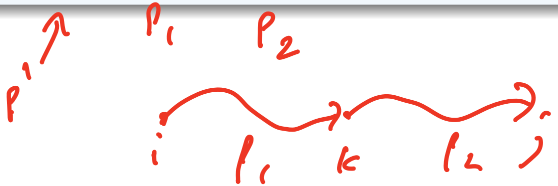

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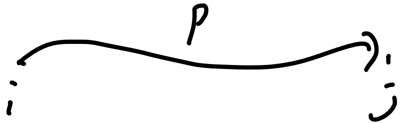


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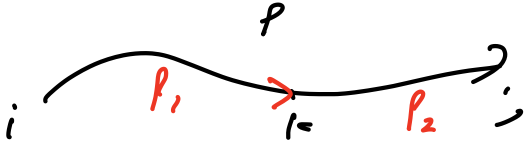


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- If $\mathbf{k}$ is an intermediate node of $\mathbf{P}$: divide $\mathbf{P}$ into $\mathbf{P}_1$ (subpath from $\mathbf{i}$ to $\mathbf{k}$) and $\mathbf{P}_2$ (subpath from $\mathbf{k}$ to $\mathbf{j}$)

$$\min(\mathbf{d}_{ij}^{k-1}, \mathbf{d}_{ik}^{k-1} + \mathbf{d}_{kj}^{k-1}) \leq \mathbf{d}_{ik}^{k-1} + \mathbf{d}_{kj}^{k-1} \leq \ell(\mathbf{P}_1) + \ell(\mathbf{P}_2) = \ell(\mathbf{P}) = \mathbf{d}_{ij}^{k}$$

# Floyd-Warshall Algorithm

Usually bottom-up, since so simple:

$\mathbf{M[i,j,0]} = \ell(\mathbf{i,j})$ for all $\mathbf{i,j} \in [\mathbf{n}]$
for($\mathbf{k} = \mathbf{1}$ to $\mathbf{n}$)
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**Correctness:** obvious for $\mathbf{k} = \mathbf{0}$. For $\mathbf{k} \geq \mathbf{1}$:

$$\begin{aligned}
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**Running Time:** $O(n^3)$

# Johnson's Algorithm

# Reweighting

Different Approach: Can we “fix” negative weights so Dijkstra from every node works?

- Time would be $\mathbf{O(n(m + n \log n)) = O(mn + n^2 \log n)}$, better than Floyd-Warshall

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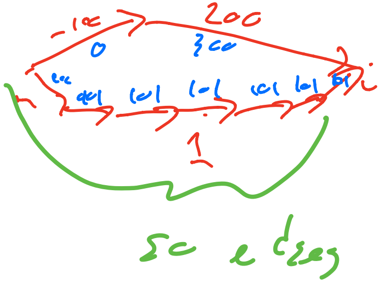

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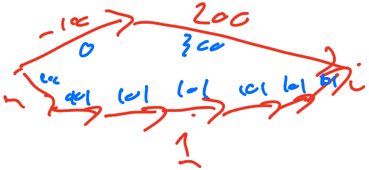

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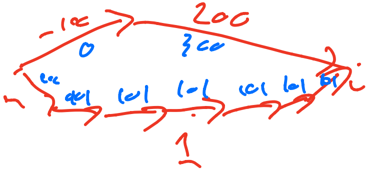


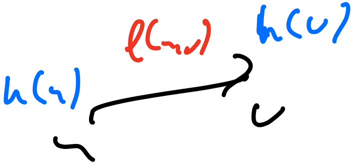

Some other kind of reweighting? Need new lengths $\hat{\ell}$ such that:

- Path $\mathbf{P}$ a shortest path under lengths $\ell$ if and only $\mathbf{P}$ a shortest path under lengths $\hat{\ell}$
- $\hat{\ell}(\mathbf{u}, \mathbf{v}) \geq \mathbf{0}$ for all $(\mathbf{u}, \mathbf{v}) \in \mathbf{E}$

# Vertex Reweighting

Neat observation: put weights at *vertices*!

- Let $\mathbf{h} : \mathbf{V} \to \mathbb{R}$ be node weights.
- Let $\ell_{\mathbf{h}}(\mathbf{u}, \mathbf{v}) = \ell(\mathbf{u}, \mathbf{v}) + \mathbf{h}(\mathbf{u}) - \mathbf{h}(\mathbf{v})$



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- Let $\ell_h(u, v) = \ell(u, v) + h(u) - h(v)$

Let $P = \langle v_0, v_1, \ldots, v_k \rangle$ be arbitrary (not necessarily shortest) path.

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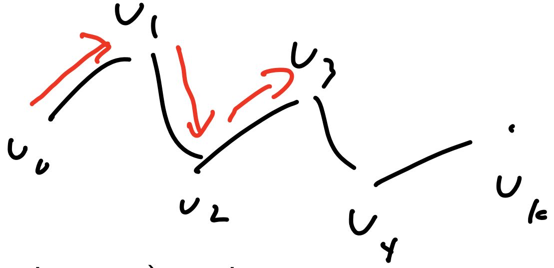


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&= \mathbf{h}(\mathbf{v_0}) - \mathbf{h}(\mathbf{v_k}) + \sum_{i=0}^{k-1} \ell(\mathbf{v_i}, \mathbf{v_{i+1}}) && \text{(telescoping)} \\
&= \ell(\mathbf{P}) + \mathbf{h}(\mathbf{v_0}) - \mathbf{h}(\mathbf{v_k})
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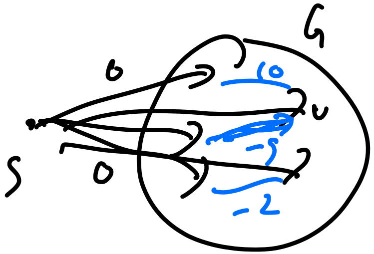
&= \ell(\mathbf{P}) + \mathbf{h}(\mathbf{v}_0) - \mathbf{h}(\mathbf{v}_k)
\end{aligned}$$

$\mathbf{h}(\mathbf{v}_0) - \mathbf{h}(\mathbf{v}_k)$ added to *every* $\mathbf{v}_0 \to \mathbf{v}_k$ path, so shortest path from $\mathbf{v}_0$ to $\mathbf{v}_k$ still shortest path!

# Making lengths nonnegative

So vertex reweighting preserves shortest paths. Find weights to make lengths nonnegative?

Add *new node* $\mathbf{s}$ to graph, edges $(\mathbf{s}, \mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}$ of length $\mathbf{0}$



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- Note $\mathbf{h(u)} \leq \mathbf{0}$ for all $\mathbf{u} \in \mathbf{V}$

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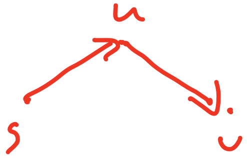


Want to show that $\ell_{\mathbf{h}}\mathbf{(u, v)} \geq \mathbf{0}$ for all edges $\mathbf{(u, v)}$.

- Triangle inequality: $\mathbf{h(v)} = \mathbf{d(s, v)} \leq \mathbf{d(s, u)} + \ell\mathbf{(u, v)} = \mathbf{h(u)} + \ell\mathbf{(u, v)}$

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# Johnson's Algorithm

- Add vertex $\mathbf{s}$ to graph, edge $\mathbf{(s, u)}$ for all $\mathbf{u \in V}$ with $\boldsymbol{\ell}\mathbf{(s, u) = 0}$
- Run Bellman-Ford from $\mathbf{s}$, set $\mathbf{h(u) = d(s, u)}$
- Remove $\mathbf{s}$, run Dijkstra from every node $\mathbf{u \in V}$ to get $\mathbf{d_h(u, v)}$ for all $\mathbf{u, v \in V}$
- If want distances, set $\mathbf{d(u, v) = d_h(u, v) - h(u) + h(v)}$ for all $\mathbf{u, v \in V}$

**Correctness:** From previous discussion.

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**Running Time:**

$O(n)$

$O(mn)$

$O(m)$ to write $\ell_h$'s

$n \cdot O(m + n \log n) = O(nm + n^2 \log n)$

$O(n^2)$

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