Lecture 15: All-Pairs Shortest Paths

Michael Dinitz

October 19, 2021
601.433/633 Introduction to Algorithms

Announcements

- HW5 due now
- HW6 due next Thursday

- Mid-Semester feedback on Campuswire!

Introduction

Setup:

- Directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$
- Length $\ell(\mathbf{x}, \mathbf{y})$ on each edge $(\mathbf{x}, \mathbf{y}) \in \mathbf{E}$
- Length of path $\mathbf{P}$ is $\ell(\mathbf{P}) = \sum_{(\mathbf{x},\mathbf{y}) \in \mathbf{P}} \ell(\mathbf{x}, \mathbf{y})$
- $\mathbf{d}(\mathbf{x}, \mathbf{y}) = \mathbf{min}_{\mathbf{x} \to \mathbf{y} \text{ paths } \mathbf{P}} \ell(\mathbf{P})$

Last time: All distances from source node $\mathbf{v} \in \mathbf{V}$.

Today: Distances between all pairs of nodes!

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- Length of path P is $\ell(P) = \sum_{(x,y)\in P} \ell(x, y)$
- $d(x, y) = \min_{x \to y \text{ paths } P} \ell(P)$

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- No negative weights: $\mathbf{n}$ runs of Dijkstra, time $\mathbf{O(n(m + n \log n))}$
- Negative weights: $\mathbf{n}$ runs of Bellman-Ford, time $\mathbf{O(nmn) = O(mn^2)}$

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Can we do better? Particularly for negative edge weights?

- Main goal today: Negative weights as fast as possible.

Floyd-Warshall Algorithm

Floyd-Warshall: A Different Dynamic Programming Approach

To simplify notation, let $\mathbf{V} = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}\}$ and $\ell(\mathbf{i}, \mathbf{j}) = \infty$ if $(\mathbf{i}, \mathbf{j}) \notin \mathbf{E}$

Bellman-Ford subproblems: length of shortest path with at most some number of edges

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New subproblems:

- Intuition: "shortest path from u to v either goes through node n, or it doesn't"
  - If it doesn't: shortest uses only first nodes in $\{1, 2, \ldots, n-1\}$.
  - If it does: consists of a path P_1 from u to n and a path P_2 from n to v, neither of which uses n (internally).

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- Subproblems: shortest path from **u** to **v** that only uses nodes in $\{1, 2, \ldots k\}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{k}$.

Formalizing Subproblems

$\mathbf{u} \to \mathbf{v}$ path $\mathbf{P}$: "intermediate nodes" are all nodes in $\mathbf{P}$ other than $\mathbf{u}, \mathbf{v}$.

$\mathbf{d_{ij}^k}$: distance from $\mathbf{i}$ to $\mathbf{j}$ using only $\mathbf{i} \to \mathbf{j}$ paths with intermediate vertices in $\{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}\}$.

- Goal: compute $\mathbf{d_{ij}^k}$ for all $\mathbf{i}, \mathbf{j}, \mathbf{k} \in [\mathbf{n}]$.
- Return $\mathbf{d_{ij}^n}$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{V}$.

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- If $\mathbf{k}$ is an intermediate node of $\mathbf{P}$: divide $\mathbf{P}$ into $\mathbf{P}_1$ (subpath from $\mathbf{i}$ to $\mathbf{k}$) and $\mathbf{P}_2$ (subpath from $\mathbf{k}$ to $\mathbf{j}$)

$$\min(\mathbf{d}_{\mathbf{ij}}^{\mathbf{k}-1}, \mathbf{d}_{\mathbf{ik}}^{\mathbf{k}-1} + \mathbf{d}_{\mathbf{kj}}^{\mathbf{k}-1}) \leq \mathbf{d}_{\mathbf{ik}}^{\mathbf{k}-1} + \mathbf{d}_{\mathbf{kj}}^{\mathbf{k}-1} \leq \ell(\mathbf{P}_1) + \ell(\mathbf{P}_2) = \ell(\mathbf{P}) = \mathbf{d}_{\mathbf{ij}}^{\mathbf{k}}$$

Floyd-Warshall Algorithm

Usually bottom-up, since so simple:

$\mathbf{M[i,j,0]} = \ell(\mathbf{i},\mathbf{j})$ for all $\mathbf{i},\mathbf{j} \in [\mathbf{n}]$
for($\mathbf{k} = \mathbf{1}$ to $\mathbf{n}$)
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Correctness: obvious for $\mathbf{k} = \mathbf{0}$. For $\mathbf{k} \geq \mathbf{1}$:

$$\begin{aligned}
\mathbf{M[i,j,k]} &= \mathbf{\min(M[i,j,k-1], M[i,k,k-1] + M[k,j,k-1])} && \text{(def of algorithm)}\\
&= \mathbf{\min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})} && \text{(induction)}\\
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Running Time: $\mathbf{O(n^3)}$

Johnson's Algorithm

Reweighting

Different Approach: Can we "fix" negative weights so Dijkstra from every node works?

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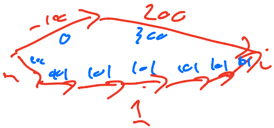

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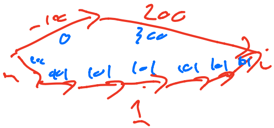

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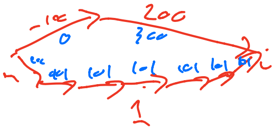

Some other kind of reweighting? Need new lengths $\hat{\ell}$ such that:

- Path $\mathbf{P}$ a shortest path under lengths ℓ if and only if $\mathbf{P}$ a shortest path under lengths $\hat{\ell}$
- $\hat{\ell}(\mathbf{u},\mathbf{v}) \geq \mathbf{0}$ for all $(\mathbf{u},\mathbf{v}) \in \mathbf{E}$

Vertex Reweighting

Neat observation: put weights at *vertices*!

- Let $\mathbf{h} : \mathbf{V} \to \mathbb{R}$ be node weights.
- Let $\ell_{\mathbf{h}}(\mathbf{u}, \mathbf{v}) = \ell(\mathbf{u}, \mathbf{v}) + \mathbf{h}(\mathbf{u}) - \mathbf{h}(\mathbf{v})$

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Let $\mathbf{P} = \langle \mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_k} \rangle$ be arbitrary (not necessarily shortest) path.

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$$\ell_{\mathbf{h}}(\mathbf{P}) = \sum_{\mathbf{i=0}}^{\mathbf{k-1}} \ell_{\mathbf{h}}(\mathbf{v_i}, \mathbf{v_{i+1}}) = \sum_{\mathbf{i=0}}^{\mathbf{k-1}} \left(\ell(\mathbf{v_i}, \mathbf{v_{i+1}}) + \mathbf{h}(\mathbf{v_i}) - \mathbf{h}(\mathbf{v_{i+1}})\right)$$

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- Let $\ell_h(u, v) = \ell(u, v) + h(u) - h(v)$

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$$\begin{aligned} \ell_h(P) &= \sum_{i=0}^{k-1} \ell_h(v_i, v_{i+1}) = \sum_{i=0}^{k-1} \left(\ell(v_i, v_{i+1}) + h(v_i) - h(v_{i+1})\right) \\ &= h(v_0) - h(v_k) + \sum_{i=0}^{k-1} \ell(v_i, v_{i+1}) \qquad \text{(telescoping)} \end{aligned}$$

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Let $\mathbf{P} = \langle \mathbf{v_0}, \mathbf{v_1}, \dots, \mathbf{v_k} \rangle$ be arbitrary (not necessarily shortest) path.

$$\begin{aligned}
\ell_{\mathbf{h}}(\mathbf{P}) &= \sum_{i=0}^{k-1} \ell_{\mathbf{h}}(\mathbf{v_i}, \mathbf{v_{i+1}}) = \sum_{i=0}^{k-1} \left(\ell(\mathbf{v_i}, \mathbf{v_{i+1}}) + \mathbf{h}(\mathbf{v_i}) - \mathbf{h}(\mathbf{v_{i+1}})\right) \\
&= \mathbf{h}(\mathbf{v_0}) - \mathbf{h}(\mathbf{v_k}) + \sum_{i=0}^{k-1} \ell(\mathbf{v_i}, \mathbf{v_{i+1}}) && \text{(telescoping)} \\
&= \ell(\mathbf{P}) + \mathbf{h}(\mathbf{v_0}) - \mathbf{h}(\mathbf{v_k})
\end{aligned}$$

Vertex Reweighting

Neat observation: put weights at *vertices*!

- Let $\mathbf{h : V \to \mathbb{R}}$ be node weights.
- Let $\ell_{\mathbf{h}}(\mathbf{u},\mathbf{v}) = \ell(\mathbf{u},\mathbf{v}) + \mathbf{h}(\mathbf{u}) - \mathbf{h}(\mathbf{v})$

Let $\mathbf{P} = \langle \mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_k} \rangle$ be arbitrary (not necessarily shortest) path.

$$\begin{aligned}
\ell_h(P) = \sum_{i=0}^{k-1} \ell_h(v_i, v_{i+1}) &= \sum_{i=0}^{k-1} \left(\ell(v_i, v_{i+1}) + h(v_i) - h(v_{i+1})\right) \\
&= h(v_0) - h(v_k) + \sum_{i=0}^{k-1} \ell(v_i, v_{i+1}) && \text{(telescoping)} \\
&= \ell(P) + h(v_0) - h(v_k)
\end{aligned}$$

$\mathbf{h}(\mathbf{v_0}) - \mathbf{h}(\mathbf{v_k})$ added to *every* $\mathbf{v_0} \to \mathbf{v_k}$ path, so shortest path from $\mathbf{v_0}$ to $\mathbf{v_k}$ still shortest path!

Making lengths nonnegative

So vertex reweighting preserves shortest paths. Find weights to make lengths nonnegative?

Add *new node* $\mathbf{s}$ to graph, edges $\mathbf{(s, v)}$ for all $\mathbf{v} \in \mathbf{V}$ of length $\mathbf{0}$

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- Run Bellman-Ford from $\mathbf{s}$, then for all $\mathbf{u} \in \mathbf{V}$ set $\mathbf{h(u)}$ to be $\mathbf{d(s, u)}$
- Note $\mathbf{h(u) \leq 0}$ for all $\mathbf{u} \in \mathbf{V}$

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- Triangle inequality: $\mathbf{h}(\mathbf{v}) = \mathbf{d}(\mathbf{s}, \mathbf{v}) \leq \mathbf{d}(\mathbf{s}, \mathbf{u}) + \ell(\mathbf{u}, \mathbf{v}) = \mathbf{h}(\mathbf{u}) + \ell(\mathbf{u}, \mathbf{v})$

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$$\ell_{\mathbf{h}}(\mathbf{u}, \mathbf{v}) = \ell(\mathbf{u}, \mathbf{v}) + \mathbf{h}(\mathbf{u}) - \mathbf{h}(\mathbf{v}) \geq \ell(\mathbf{u}, \mathbf{v}) + \mathbf{h}(\mathbf{u}) - (\mathbf{h}(\mathbf{u}) + \ell(\mathbf{u}, \mathbf{v})) = \mathbf{0}$$

Johnson's Algorithm

- Add vertex $\mathbf{s}$ to graph, edge $(\mathbf{s}, \mathbf{u})$ for all $\mathbf{u} \in \mathbf{V}$ with $\ell(\mathbf{s}, \mathbf{u}) = \mathbf{0}$
- Run Bellman-Ford from $\mathbf{s}$, set $\mathbf{h}(\mathbf{u}) = \mathbf{d}(\mathbf{s}, \mathbf{u})$
- Remove $\mathbf{s}$, run Dijkstra from every node $\mathbf{u} \in \mathbf{V}$ to get $\mathbf{d}_\mathbf{h}(\mathbf{u}, \mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$
- If want distances, set $\mathbf{d}(\mathbf{u}, \mathbf{v}) = \mathbf{d}_\mathbf{h}(\mathbf{u}, \mathbf{v}) - \mathbf{h}(\mathbf{u}) + \mathbf{h}(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$

Correctness: From previous discussion.

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Correctness: From previous discussion.

Running Time:

Johnson's Algorithm

- Add vertex $\mathbf{s}$ to graph, edge $\mathbf{(s, u)}$ for all $\mathbf{u} \in \mathbf{V}$ with $\ell\mathbf{(s, u) = 0}$
- Run Bellman-Ford from $\mathbf{s}$, set $\mathbf{h(u) = d(s, u)}$
- Remove $\mathbf{s}$, run Dijkstra from every node $\mathbf{u} \in \mathbf{V}$ to get $\mathbf{d_h(u, v)}$ for all $\mathbf{u, v} \in \mathbf{V}$
- If want distances, set $\mathbf{d(u, v) = d_h(u, v) - h(u) + h(v)}$ for all $\mathbf{u, v} \in \mathbf{V}$

Correctness: From previous discussion.

Running Time: $\mathbf{O(n) + O(mn) + O(n(m + n \log n)) = O(mn + n^2 \log n)}$