Lecture 10: Universal and Perfect Hashing

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601.433/633 Introduction to Algorithms

Introduction

Another approach to dictionaries (insert, lookup, delete): hashing

- Can improve operations to $\mathbf{O(1)}$, but with many caveats!

Should have seen some discussion of hashing in data structures. Also in CLRS.

- Separate chaining vs. open addressing

Today: discussion of caveats, more advanced versions of hashing (universal and perfect)

Hashing Basics

- Keys from universe $\mathbf{U}$ (think very large)
- Set $\mathbf{S} \subseteq \mathbf{U}$ of keys we actually care about (think relatively small). $|\mathbf{S}| = \mathbf{N}$.
- Hash table $\mathbf{A}$ (array) of size $\mathbf{M}$.
- Hash function $\mathbf{h : U \to [M]}$
  - $\mathbf{[M] = \{1, 2, \ldots, M\}}$
- Idea: store $\mathbf{x}$ in $\mathbf{A[h(x)]}$

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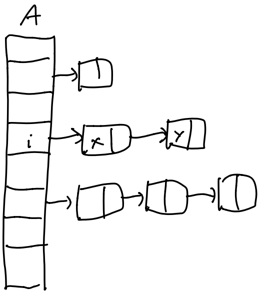
  - $[\mathbf{M}] = \{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{M}\}$
- Idea: store $\mathbf{x}$ in $\mathbf{A}[\mathbf{h}(\mathbf{x})]$

One more component: *collision resolution*

- Today: *separate chaining*
- $\mathbf{A}[\mathbf{i}]$ is a linked list containing all $\mathbf{x}$ inserted where $\mathbf{h}(\mathbf{x}) = \mathbf{i}$.



Dictionary Operations

Lookup($\mathbf{x}$): Walk down the list at $\mathbf{A}[\mathbf{h}(\mathbf{x})]$ until we find $\mathbf{x}$ (or walk to the end of the list)

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- Small $\mathbf{M}$. Ideally, $\mathbf{M = O(N)}$.
- $\mathbf{h}$ fast to compute.

Bad News

Theorem

For any hash function $\mathbf{h}$, *if* $|\mathbf{U}| \geq (\mathbf{N}-\mathbf{1})\mathbf{M}+\mathbf{1}$, *then there exists a set* $\mathbf{S}$ *of* $\mathbf{N}$ *elements that all hash to the same location.*

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- Option 2: Randomness! *Random function* $\mathbf{h}:\mathbf{U}\to[\mathbf{M}]$
  - For each $\mathbf{x}\in\mathbf{U}$, choose $\mathbf{y}\in[\mathbf{M}]$ uniformly at random and set $\mathbf{h}(\mathbf{x})=\mathbf{y}$.
  - Hopefully good behavior in expectation.

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  - Hopefully good behavior in expectation.
  - Problem: How can we store/remember/create $\mathbf{h}$?

Universal Hashing

Definition

A probability distribution $\mathbf{H}$ over hash functions $\{\mathbf{h} : \mathbf{U} \to [\mathbf{M}]\}$ is *universal* if

$$\Pr_{\mathbf{h} \sim \mathbf{H}}[\mathbf{h}(\mathbf{x}) = \mathbf{h}(\mathbf{y})] \leq \mathbf{1}/\mathbf{M}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{U}$ with $\mathbf{x} \neq \mathbf{y}$.

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If $\mathbf{H}$ is universal, then for every set $\mathbf{S} \subseteq \mathbf{U}$ with $|\mathbf{S}| = \mathbf{N}$ and for every $\mathbf{x} \in \mathbf{U}$, the expected number of collisions (when we draw $\mathbf{h}$ from $\mathbf{H}$) between $\mathbf{x}$ and elements of $\mathbf{S}$ is at most $\mathbf{N}/\mathbf{M}$.

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So Lookup($\mathbf{x}$) and Delete($\mathbf{x}$) have expected time $\mathbf{O}(\mathbf{N}/\mathbf{M})$.
$\implies$ If $\mathbf{M} = \mathbf{\Omega}(\mathbf{N})$, operations in $\mathbf{O}(\mathbf{1})$ time!

Main Proof

Theorem

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Proof.

Let $\mathbf{C}_{\mathbf{xy}} = \begin{cases} \mathbf{1} & \text{if } \mathbf{h}(\mathbf{x}) = \mathbf{h}(\mathbf{y}) \\ \mathbf{0} & \text{otherwise} \end{cases}$

$$\implies \mathbf{E}[\mathbf{C}_{\mathbf{xy}}] = \Pr_{\mathbf{h} \sim \mathbf{H}}[\mathbf{h}(\mathbf{x}) = \mathbf{h}(\mathbf{y})] \leq \mathbf{1}/\mathbf{M}$$

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Let $\mathbf{C_{xy}} = \begin{cases} \mathbf{1} & \text{if } \mathbf{h(x) = h(y)} \\ \mathbf{0} & \text{otherwise} \end{cases}$

$$\Longrightarrow \mathbf{E[C_{xy}]} = \Pr_{\mathbf{h \sim H}}[\mathbf{h(x) = h(y)}] \leq \mathbf{1/M}$$

Number of collisions between $\mathbf{x}$ and $\mathbf{S}$ is exactly $\sum_{\mathbf{y \in S}} \mathbf{C_{xy}}$

$$\Longrightarrow \mathbf{E}\left[\sum_{\mathbf{y \in S}} \mathbf{C_{xy}}\right] = \sum_{\mathbf{y \in S}} \mathbf{E}\left[\mathbf{C_{xy}}\right] \leq \sum_{\mathbf{y \in S}} \frac{\mathbf{1}}{\mathbf{M}} = \mathbf{N/M}$$

□

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If $\mathbf{H}$ is universal, then for any sequence of $\mathbf{L}$ insert, lookup, and delete operations in which there are at most $\mathbf{O(M)}$ elements in the system at any time, the expected total cost of the whole sequence is only $\mathbf{O(L)}$ (assuming $\mathbf{h}$ takes constant time to compute).

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So universal distributions are great. Can we construct them?

Universal Hash Families

Definition

If $\mathbf{H}$ is universal and is a uniform distribution over a set of functions $\{\mathbf{h_1}, \mathbf{h_2}, \ldots\}$, then that set is called a *universal hash family*.

Often use $\mathbf{H}$ to refer to both set of functions and uniform distribution over it.

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Notation:

- $\mathbf{U} = \{\mathbf{0}, \mathbf{1}\}^{\mathbf{u}}$ (so $|\mathbf{U}| = \mathbf{2}^{\mathbf{u}}$)
- $\mathbf{M} = \mathbf{2}^{\mathbf{b}}$, so an index to $\mathbf{A}$ is an element of $\{\mathbf{0}, \mathbf{1}\}^{\mathbf{b}}$

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$$\begin{array}{ccccc} h & & x & & h(x) \\ \boxed{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}} & & \boxed{\begin{matrix} 1 \\ 0 \\ 1 \\ 0 \end{matrix}} & = & \boxed{\begin{matrix} 1 \\ 1 \\ 0 \end{matrix}} \end{array}$$

Construction: $\mathbf{H} = \{\mathbf{0},\mathbf{1}\}^{\mathbf{b}\times\mathbf{u}}$, i.e., $\mathbf{H}$ is all $\mathbf{b}\times\mathbf{u}$ binary matrices

- Each $\mathbf{h} \in \mathbf{H}$ is a (linear) function from $\mathbf{U}$ to $[\mathbf{M}]$: $\mathbf{h}(\mathbf{x}) = \mathbf{hx} \in \{\mathbf{0},\mathbf{1}\}^{\mathbf{b}}$ (all operations mod 2)

Universality

Theorem

$\mathbf{H}$ *is a universal hash family:* $\mathbf{Pr}_{\mathbf{h \sim H}}[\mathbf{h(x) = h(y)}] = \mathbf{1/M}$ *for all* $\mathbf{x \neq y \in \{0,1\}^u}$.

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Since $\mathbf{x} \neq \mathbf{y}$, there is $\mathbf{i}$ s.t. $\mathbf{x}_\mathbf{i} \neq \mathbf{y}_\mathbf{i}$. WLOG, $\mathbf{x}_\mathbf{i} = \mathbf{0}$ and $\mathbf{y}_\mathbf{i} = \mathbf{1}$.

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Draw all entries of $\mathbf{h}$ *except* for $\mathbf{h}^\mathbf{i}$. Let $\mathbf{h}' = \mathbf{h}$ with $\mathbf{h}^\mathbf{i}$ all $\mathbf{0}$'s

- $\mathbf{h}(\mathbf{x}) = \mathbf{h}'(\mathbf{x})$ already fixed.

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- If $\mathbf{h}(\mathbf{y}) = \mathbf{h}(\mathbf{x})$, then $\mathbf{h^i}$ must equal $\mathbf{h}(\mathbf{x}) - \mathbf{h}'(\mathbf{y})$
- Happens with probability exactly $\mathbf{1/2^b} = \mathbf{1/M}$ □

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Can we do better with hashing? Yes, through universal hashing!

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Fix $\mathbf{x}, \mathbf{y} \in \mathbf{S}$ with $\mathbf{x} \neq \mathbf{y}$.

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Fix $\mathbf{x, y \in S}$ with $\mathbf{x \neq y}$.

$\mathbf{Pr_{h\sim H}[h(x) = h(y)] \leq 1/M = 1/N^2}$ by universality.

$$\Pr_{h\sim H}[\exists \text{ collision in } S] \leq \sum_{\substack{x,y\in S \\ x\neq y}} \Pr_{h\sim H}[h(x) = h(y)] \leq \sum_{\substack{x,y\in S \\ x\neq y}} \frac{1}{N^2}$$

$$= \binom{N}{2}\frac{1}{N^2} = \frac{N(N-1)}{2}\frac{1}{N^2} \leq \frac{1}{2}$$

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$$\begin{aligned}\Pr_{h\sim H}[\exists \text{ collision in } S] &\leq \sum_{\substack{x,y\in S\\ x\neq y}} \Pr_{h\sim H}[h(x) = h(y)] \leq \sum_{\substack{x,y\in S\\ x\neq y}} \frac{1}{N^2}\\ &= \binom{N}{2}\frac{1}{N^2} = \frac{N(N-1)}{2}\frac{1}{N^2} \leq \frac{1}{2}\end{aligned}$$

□

So keep sampling $\mathbf{h} \sim \mathbf{H}$ until get one with no collisions!

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Let $\mathbf{S_i} = \{\mathbf{x} \in \mathbf{S} : \mathbf{h(x)} = \mathbf{i}\}$ and let $\mathbf{n_i} = |\mathbf{S_i}|$

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Starting approach: set $\mathbf{M} = \mathbf{N}$, use a universal hash family $\mathbf{H}$. Draw $\mathbf{h} \sim \mathbf{H}$.

- Will have collisions. Need to do something other than chaining.

Let $\mathbf{S_i} = \{\mathbf{x} \in \mathbf{S} : \mathbf{h(x)} = \mathbf{i}\}$ and let $\mathbf{n_i} = |\mathbf{S_i}|$

- Use another hash table for $\mathbf{S_i}$!
- Use Method 1: $\mathbf{O(n_i^2)}$-size perfect hashing of $\mathbf{S_i}$.
  - Let $\mathbf{h_i : U \to [n_i^2]}$ be hash function for $\mathbf{S_i}$, and $\mathbf{A_i}$ be table (pointer from $\mathbf{A[i]}$)

Method 2

$\mathbf{M} = \mathbf{N}^2$ is pretty big!

- Only storing $\mathbf{N}$ things, and know them ahead of time
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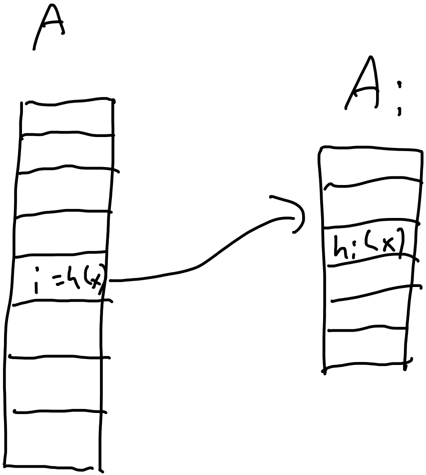
Lookup($\mathbf{x}$): Look in linked list at $\mathbf{A_{h(x)}[h_{h(x)}(x)]}$

Picture



Analysis

Lookup time: by analysis of Method 1, no collisions in second level.

$\Longrightarrow$ Lookup time $\mathbf{O(1)}$

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Size: $\mathbf{O(N + \sum_{i=1}^{N} n_i^2)}$

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Theorem

Let $\mathbf{H}$ *be universal onto a table of size* $\mathbf{N}$*. Then*

$$\Pr_{h \sim H}\left[\sum_{i=1}^{N} n_i^2 > 4N\right] < 1/2.$$

So like with method 1: keep drawing $\mathbf{h \sim H}$ until $\mathbf{\sum_{i=1}^{N} n_i^2 \le 4N}$

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Prove that $E\left[\sum_{i=1}^{N} n_i^2\right] \leq 2N$.

- Implies theorem by Markov's inequality
  - $\Pr[X > 2E[X]] \leq 1/2$ for nonnegative random variables X.

Proof

Observation: $\sum_{i=1}^{N} n_i^2$ is exactly number of *ordered* pairs that collide, including self-collisions

- Example: If $S_i = \{a, b, c\}$ then $n_i^2 = 9$. Ordered colliding pairs: $(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)$

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$$\begin{aligned}
E\left[\sum_{i=1}^{N} n_i^2\right] &= E\left[\sum_{x \in S}\sum_{y \in S} C_{xy}\right] \\
&= N + \sum_{x \in S}\sum_{y \in S: y \neq x} E\left[C_{xy}\right] && \text{(linearity of expectations)} \\
&\leq N + \frac{N(N-1)}{M} && \text{(definition of universal)} \\
&< 2N && (\text{since } M = N)
\end{aligned}$$