601.435 / 601.635 Approximation Algorithms **Spring 2026**
Homework #3 **Due:** March 12, 2026, 9:00am

Reminder: you may work in groups of up to three people, but must write up solutions entirely on your own. Collaboration is limited to discussing the problems – you may not look at, compare, reuse, etc. any text from anyone else in the class. Please include your list of collaborators on the first page of your submission. Many of these problems have solutions which can be found on the internet – please don't look. You can of course use the internet (including the links provided on the course webpage) as a learning tool, but don't go looking for solutions.

Please include proofs with all of your answers, unless stated otherwise.

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1 SONET ring loading (50 points)

The following problem, known as the *SONET ring loading problem*, is a classical problem in telecommunications networks. We are given an undirected cycle on n nodes, numbered 0 through $n-1$ clockwise around the cycle. We are also given a set T of calls, where each call is a pair (i,j) originating at node i and destined to node j. The call can be routed either clockwise or counterclockwise through the cycle. The objective is to route the calls so as to minimize the maximum load on the network: the load L_i on link $\{i,(i+1) \pmod n\}$ is the number of calls routed through the link (in either direction), and the maximum load is $\max_{0\le i\le n-1} L_i$.

(a) (20 points) Write an LP relaxation for this problem, and use it to give a 2-approximation algorithm by using *deterministic* rounding on the LP.

Solution. Let E be the edges in the cycle. For each $(i,j)\in T$, let A_{ij} denote the edges in the clockwise path from i to j and let $B_{ij}=E\setminus A_{ij}$ be the edges in the counterclockwise path from i to j. For each $(i,j)\in T$ we'll have a variable x_{ij}. We also have a single variable W. Consider the following LP relaxation.

$$\begin{aligned} \min \quad & W \\ \text{subject to} \quad & \sum_{(i,j)\in T: e\in A_{ij}} x_{ij} + \sum_{(i,j)\in T: e\in B_{ij}} (1-x_{ij}) \le W && \forall e\in E \\ & 0\le x_{ij}\le 1 && \forall i\in V \\ & W\ge 0 \end{aligned}$$

It is easy to see that this is a valid relaxation, i.e., that $LP\le OPT$. To see this, consider some optimal solution. For some $(i,j)\in T$, if the optimal solution routes the call clockwise then we set $x_{ij}=1$, and otherwise we set $x_{ij}=0$. Then $\sum_{(i,j)\in T:e\in A_{ij}} x_{ij} + \sum_{(i,j)\in T:e\in B_{ij}}(1-x_{ij})$ is equal to the load in edge e, and hence if we set W to the maximum load then this is a feasible LP solution of cost OPT. Hence $LP\le OPT$.

So to give a 2-approximation algorithm we first solve the LP (in polynomial time, since the LP has polynomial size) to get an optimal solution (x,W). Then we round this solution by routing $(i,j)\in T$ clockwise if $x_{ij}\ge 1/2$ and otherwise routing (i,j) counterclockwise. Let $Y\subseteq T$ be the calls that we route clockwise, and let Z be the calls that we route counterclockwise. Let e be the edge which has the maximum load under this solution. Then the load on e is

$$\begin{aligned}
\sum_{(i,j)\in Y: e\in A_{ij}} 1 + \sum_{(i,j)\in Z: e\in B_{ij}} 1 &= \sum_{(i,j)\in T: x_{ij}\geq 1/2 \wedge e\in A_{ij}} 1 + \sum_{(i,j)\in T: x_{ij}< 1/2 \wedge e\in B_{ij}} 1 \\
&\leq \sum_{(i,j)\in T: x_{ij}\geq 1/2 \wedge e\in A_{ij}} 2x_{ij} + \sum_{(i,j)\in T: x_{ij}< 1/2 \wedge e\in B_{ij}} 2(1-x_{ij}) \\
&\leq \sum_{(i,j)\in T: e\in A_{ij}} 2x_{ij} + \sum_{(i,j)\in T: e\in B_{ij}} 2(1-x_{ij}) \\
&= 2\left(\sum_{(i,j)\in T: e\in A_{ij}} x_{ij} + \sum_{(i,j)\in T: e\in B_{ij}} (1-x_{ij})\right) \\
&\leq 2W \\
&\leq 2\cdot OPT
\end{aligned}$$

(b) (10 points) Prove that this is tight by proving that the integrality gap of your LP relaxation is at least 2.

Solution. Suppose that there is a single call (i,j) (it does not matter what i and j are as long as $i \neq j$). Clearly the optimal solution has cost 1, since this call has to be routed either clockwise or counterclockwise and so some edge has load 1. On the other hand, the LP can set $x_{ij} = 1/2$ and $W = 1/2$. This is a feasible LP solution with objective 1/2. Hence $OPT/LP \geq 2$ for this instance.

(c) (20 points) Now suppose that we are also given a positive capacity $c_e \in \mathbb{R}^+$ for each edge e in the cycle and a demand $d_{(i,j)} \in \mathbb{R}^+$ for each call $(i,j) \in T$. A natural generalization of the problem would be to define the load on an edge $\{i, (i+1) \pmod n\}$ to be the sum of the demands of the calls routed through the link divided by the capacity of the link, and then the objective function is to minimize the max load (as before). Note that if all capacities and demands are 1 then this is exactly the SONET ring loading problem. Give a (deterministic) 2-approximation algorithm for this problem.

Solution. We modify the original LP as follows.

$$\begin{aligned}
\min \quad & W \\
\text{subject to} \quad & \sum_{(i,j)\in T: e\in A_{ij}} d_{(i,j)} x_{ij} + \sum_{(i,j)\in T: e\in B_{ij}} d_{(i,j)}(1-x_{ij}) \leq W c(e) && \forall e\in E \\
& 0 \leq x_{ij} \leq 1 && \forall i \in V \\
& W \geq 0
\end{aligned}$$

As before, we can argue that this is a valid LP relaxation. Consider some optimal solution. For some $(i,j)\in T$, if the optimal solution routes the call clockwise then we set $x_{ij} = 1$, and otherwise we set $x_{ij} = 0$. Then

$$\frac{\sum_{(i,j)\in T: e\in A_{ij}} d_{(i,j)} x_{ij} + \sum_{(i,j)\in T: e\in B_{ij}} d_{(i,j)}(1 - x_{ij})}{c(e)}$$

is equal to the load in edge e, and hence if we set W to the maximum load then this is a feasible LP solution of cost OPT. Hence $LP \leq OPT$.

So to give a 2-approximation algorithm we first solve the LP (in polynomial time, since the LP has polynomial size) to get an optimal solution (x, W). Then we round this solution by routing $(i, j) \in T$ clockwise if $x_{ij} \geq 1/2$ and otherwise routing (i, j) counterclockwise. Let $Y \subseteq T$ be the calls that we route clockwise, and let Z be the calls that we route counterclockwise. Let e be the edge which has the maximum load under this solution. Then the load on e is

$$\begin{aligned}
&\frac{1}{c(e)}\left(\sum_{(i,j)\in Y: e\in A_{ij}} d_{(i,j)} + \sum_{(i,j)\in Z: e\in B_{ij}} d_{(i,j)}\right)\\
&= \frac{1}{c(e)}\left(\sum_{(i,j)\in T: x_{ij}\geq 1/2 \wedge e\in A_{ij}} d_{(i,j)} + \sum_{(i,j)\in T: x_{ij}< 1/2 \wedge e\in B_{ij}} d_{(i,j)}\right)\\
&\leq \frac{1}{c(e)}\left(\sum_{(i,j)\in T: x_{ij}\geq 1/2 \wedge e\in A_{ij}} 2x_{ij} d_{(i,j)} + \sum_{(i,j)\in T: x_{ij}< 1/2 \wedge e\in B_{ij}} 2(1 - x_{ij}) d_{(i,j)}\right)\\
&\leq \frac{1}{c(e)}\left(\sum_{(i,j)\in T: e\in A_{ij}} 2x_{ij} d_{(i,j)} + \sum_{(i,j)\in T: e\in B_{ij}} 2(1 - x_{ij}) d_{(i,j)}\right)\\
&= \frac{2}{c(e)}\left(\sum_{(i,j)\in T: e\in A_{ij}} x_{ij} + \sum_{(i,j)\in T: e\in B_{ij}} (1 - x_{ij})\right)\\
&\leq 2W\\
&\leq 2 \cdot OPT
\end{aligned}$$

2 Maximum Directed Cut (Exercises 5.3, 5.6) (50 points)

In the *maximum directed cut* problem (known as MAX DICUT), the input is a directed graph $G = (V, E)$ and for each edge $(i, j \in E)$ there is a nonnegative weight $w_{ij} \geq 0$. The goal is to partition V into two sets U and $W = V \setminus U$ in order to maximize the total weight of the edges going from U to W (that is, edges (i, j) with $i \in U$ and $j \in W$).

(a) (15 points) Give a simple randomized 4-approximation to this problem (no LPs necessary).

Solution: For each vertex i, add it to U independently with probability $1/2$. Then every edge goes from U to W with probability $1/4$, so by linearity of expectations the expected weight from U to W is $1/4$ of the total edge weight. Since the total edge weight is an upper bound on OPT, this yields a 4-approximation.

(b) (15 points) Prove that the following ILP is an exact formulation: any cut gives an ILP solution with at least as large value, and any ILP solution gives a cut with at least as large value.

$$\begin{array}{lll} \max & \displaystyle\sum_{\{i,j\}\in E} w_{ij} z_{ij} & \\ \text{subject to} & z_{ij} \le x_i & \forall (i,j)\in E \\ & z_{ij} \le 1 - x_j & \forall (i,j)\in E \\ & 0 \le z_{ij} \le 1 & \forall (i,j)\in E \\ & 0 \le x_i \le 1 & \forall i \in V \\ & x_i \in \mathbb{Z} & \forall i \in V \\ & z_{ij} \in \mathbb{Z} & \forall (i,j)\in E \end{array}$$

Solution. Let (U, W) be a cut. For each $i \in V$, set $x_i = 1$ if $i \in U$ and $x_i = 0$ if $i \in W$. For each edge $(i,j) \in E$, set $z_{ij} = 1$ if $i \in U$ and $j \in W$, and set $z_{ij} = 0$ otherwise. Then clearly $\sum_{(i,j)\in E} w_{ij} z_{ij} = \sum_{(i,j)\in E : i \in U, j \in W} w_{ij}$, and hence the objective function of the ILP is equal to the weight of the cut. So it remains to prove that all constraints are satisfied. Since $z_{ij} = 1$ only if $i \in U$, we get that $z_{ij} \le x_i$. Similarly, since $z_{ij} = 1$ only if $j \in W$ we get that $z_{ij} \le 1 - x_j$. Hence all constraints are satisfied.

Now let (x, z) be a solution to the ILP. Let $U = \{i \in V : x_i = 1\}$, and let $W = V \setminus U = \{j \in V : x_j = 0\}$. The LP constraints imply that $z_{ij} = 1$ only if $i \in U$ and $j \notin U$. Hence the weight of the cut is at least as large as the value of the ILP.

(c) (20 points) Consider a randomized rounding algorithm which works as follows: we first solve the LP relaxation of the ILP from part (b), and then for each vertex $i \in V$, it adds i to U with probability $\frac{1}{4} + \frac{x_i}{2}$. Prove that this gives a 2-approximation to MAX DICUT.

Solution: Consider edge $e = (i,j)$. The probability that e is included in the directed cut is

$$\begin{aligned} &\left(\frac{1}{4} + \frac{x_i}{2}\right)\left(1 - \left(\frac{1}{4} + \frac{x_j}{2}\right)\right) = \left(\frac{1}{4} + \frac{1}{2}x_i\right)\left(\frac{1}{4} + \frac{1}{2}(1 - x_j)\right) \\ &\ge \left(\frac{1}{4} + \frac{1}{2}z_{ij}\right)\left(\frac{1}{4} + \frac{1}{2}z_{ij}\right) = \left(\frac{1}{4} + \frac{1}{2}z_{ij}\right)^2 \\ &= \frac{1}{4}z_{ij}^2 + \frac{1}{4}z_{ij} + \frac{1}{16} \ge z_{ij}/2 \end{aligned}$$

In these inequalities we use the inequalities from the LP to go from the x variables to the z variables, and then basic algebra for the final bound. Hence the expected total weight across the cut is at least $\sum_{\{i,j\}\in E} w_{ij} z_{ij}/2 = (1/2)\sum_{\{i,j\}\in E} w_{ij} z_{ij} \ge 1/2 \cdot OPT$, since we know from part b that the optimal fractional solution to this LP is at least OPT.