Min-Makespan on Identical Parallel Machines

Input: Jobs \( J \) (\( |J| = n \))
- Machines \( M \) (\( |M| = k \))
- Processing times \( p: J \rightarrow \mathbb{N} \)

Feasible Solution: \( \pi : J \rightarrow M \)

Objective: \( \text{min makespan} = \min \ max_{m \in M} \sum_{j \in J: \pi(j) = m} p(j) \)

The greedy is a 2-approximation.

Want to do better!
Idea: what if we have a “guess” $T$ for $\text{OPT}$?

Def: A $(1+\varepsilon)$-relaxed decision procedure is an algorithm which, given input $\pi$ and $T$:

1) If $T \geq \text{OPT}$, returns a solution with makespan $\leq (1+\varepsilon)T$

2) If $T < \text{OPT}$, either returns “false” or returns a solution with makespan $\leq (1+\varepsilon) \cdot \text{OPT}$

Let $P = \sum_{i \in J} p(i)$. Then $1 \leq \text{OPT} \leq P$

Idea: do binary search to find value of $T$ s.t. procedure returns a solution on $T$, “false” on $T-1$

$\Rightarrow T \leq \text{OPT}$
$\Rightarrow$ solution has makespan $\leq (1+\varepsilon) \cdot \text{OPT}$

Takes $O(\log P)$ iterations: polytime!

So just want to design such a procedure, given guess $T$.

Def: $J_{\text{small}} = \{ i \in J : p(i) \leq \varepsilon T \}$
$J_{\text{large}} = \{ i \in J : p(i) > \varepsilon T \}$
Then given schedule for Jorge with makespan \( \leq (1+\varepsilon)T \), we can find a schedule for J with makespan
\[
\leq (1+\varepsilon) \max(T, \text{OPT})
\]

Proof:

Start with schedule for Jorge.

Next, on J:

- in arbitrary order, add job to least-loaded machine

Consider machine \( m \).

- \( \text{Case 1} : m \) has no small jobs.
  \[ \Rightarrow \text{load} \leq (1+\varepsilon)T \]
- \( \text{Case 2} : m \) has \( \geq 1 \) small job

Let \( j \) be last small job assigned to \( m \)

\[ \exists p(j) \leq \varepsilon\cdot T \]

\[ \text{load on } m \text{ just before } \leq \frac{p}{k} \leq \text{OPT} \]

\[ \exists \text{total load } \leq \text{OPT} + \varepsilon T \leq (1+\varepsilon) \max(T, \text{OPT}) \]
So just need to schedule $J_{true}$ with makespan $\leq (1+\varepsilon)T$

Let $b = \lceil \frac{1}{\varepsilon} \rceil$, so $\frac{1}{b} \leq \varepsilon$

**Def:** Let $p'(i) = \left\lfloor \frac{p(i)}{b^2} \right\rfloor \cdot \frac{T}{b^2}$ ("rounded instance")

$p'$ just $p$ rounded down to multiple of $\frac{T}{b^2}$

$\Rightarrow p'(i) \leq p(i) \leq p'(i) + \frac{T}{b^2}$

$p'(i) = k \frac{T}{b^2}$ for some $k \in \{b, b+1, \ldots, b^2\}$ (since $i \in J_{true}$)

For schedule $\overline{I}$, let $m(\overline{I})$ be makespan in original
$m_r(\overline{I})$ makespan in rounded.

**Lemma 1:** Let $I$ schedule. Then $m_r(\overline{I}) \leq m(\overline{I})$

**Proof:**

$p'(i) \leq p(i) \quad \forall i \in \overline{I}$

**Lemma 2:** Let $I$ schedule with $m_r(\overline{I}) \leq T$. Then
$m(\overline{I}) \leq (1+\varepsilon)T$
Let $m$ some machine
since $p'(i) \geq \frac{T}{b^2}$, $|\mathcal{I}^m| \leq b$

$$\sum_{i \in \mathcal{I}^m} p(i) \leq \sum_{i \in \mathcal{I}^m} \left( p'(i) + \frac{T}{b^2} \right)$$

$$= \sum_{i \in \mathcal{I}^m} p'(i) + \sum_{i \in \mathcal{I}^m} \frac{T}{b^2} \leq T + b \cdot \frac{T}{b^2} \leq T + \frac{T}{b} \leq T + \varepsilon T = (1+\varepsilon)T$$

Suppose we had an algorithm to find some $\mathcal{I}'$ with $m(\mathcal{I}') \leq T$ if one exists

If $T \geq \text{OPT}$:

\exists \mathcal{I}' with $m(\mathcal{I}') \leq T \Rightarrow m(\mathcal{I}') \leq T$ \hspace{1cm} (Lemma 1)

\Rightarrow \exists $\text{schedule } w/ \text{ makespan } \leq T$ in rounded

\Rightarrow $\text{alg will find some } \mathcal{I}' \text{ with } m(\mathcal{I}') \leq T$

$\Rightarrow m(\mathcal{I}') \leq (1+\varepsilon)T$ \hspace{1cm} (Lemma 2)
If \( T < \text{OPT} \):

1. If alg returns \( T \), have

\[ m(T) \leq (1+\varepsilon) T \quad \text{(Lemma 2)} \]

So just need to find \( T \) with \( m(T) \leq T \) if one exists!

**Def:** A configuration is a tuple \((a_1, a_2, \ldots, a_b)\)

such that:

1. Each \( a_i \in \{0, 1, \ldots, b\} \)
2. \( b \sum_{i=1}^{b} \frac{i^2}{a_i} \leq T \)

**Idea:** Look at jobs assigned to some machine, let \( a_i \) be

with \( p_i(s) = i \cdot \frac{T}{b^2} \)

\( \Rightarrow \) get a configuration
Let $\mathbb{C}(T)$ be set of all configurations

$|\mathbb{C}(T)| \leq (b+1)^{b^2-b} = (b)^{b^2}$

Dynamic Programming: given (bounded) jobs, how many machines are necessary?

More formally:

$$f(n_a, n_{a_1}, \ldots, n_{a_k}) = \min m \text{ s.t. can schedule } n_i \text{ jobs of length } i \cdot \frac{T}{b^2} \text{ w/ makespan } \leq T$$

$$f(\emptyset) = 0$$

$$f(n_a, \ldots, n_{a_k}) = 1 + \min_{a \in \mathbb{C}(T)} f(n_{a-a_b}, n_{a_{b+1}}-a_{b+1}, \ldots, n_{a_k}-a_{b+k})$$

Time (table entry): $|\mathbb{C}(T)| = b^{b^2}$

# table entries: $\leq n^{b^2}$
(\text{total \ time} \leq n^2 = n)

Once table filled out, check entry corresponding to Jia (rounded)

If \( m \): no \( \varpi \) with \( m(\varpi) \leq T \)

else \( \varpi \) with \( m(\varpi) \leq T \)

\( \Rightarrow \) return false

If \( \leq m \): Return \( \varpi \) with \( m(\varpi) \leq T \)