

Knapsack:

- Input:
- Items $[n]$
 - profits $p: [n] \rightarrow \mathbb{N}$
 - sizes $s: [n] \rightarrow \mathbb{N}$
 - integer $k \in \mathbb{N}$

Feasible solution: $I \subseteq [n]$ s.t. $s(I) = \sum_{i \in I} s(i) \leq k$

Objective: $\max p(I) = \sum_{i \in I} p(i)$

WLOG: $s(i) \leq k \quad \forall i$

Sort: WLOG, assume $\frac{p(i)}{s(i)} \geq \frac{p(i+1)}{s(i+1)} \quad \forall i \in [n-1]$

Greedy:

- Add items until can't anymore

Bad!

- $s(1) = 1 \quad p(1) = 2$
 - $s(2) = k \quad p(2) = k$
- $\Rightarrow \sqrt{2}(k)$ -approx

Improved greedy:

- Let i be first element we can't add:

$$\sum_{j=1}^i s(j) > k, \text{ but } \sum_{j=1}^{i-1} s(j) \leq k$$

- Return better of $[i-1]$, $\{i\}$

Thm: 2-approximation

PF:

Since greedy, know $\frac{\sum_{j=1}^i p(j)}{\sum_{j=1}^i s(j)} \geq \frac{OPT}{k}$ (greedy has better "bang for the buck")

$$\Rightarrow OPT \leq \frac{k}{\sum_{j=1}^i s(j)} \cdot \sum_{j=1}^i p(j) \leq \sum_{j=1}^i p(j)$$

$$\leq \sum_{j=1}^{i-1} p(j) + p(i)$$

$$\leq 2 \max \left(\sum_{j=1}^{i-1} p(j), p(i) \right)$$

$$= 2 \cdot ALG$$

$$\Rightarrow ALG \geq \frac{1}{2} \cdot OPT$$

Dynamic Programming:

$$M = \max_{i \in (n)} p(i) \quad \Rightarrow \text{OPT} \leq nM$$

Subproblems:

- What size is necessary to achieve some profit?

$\forall i \in (n), 0 \leq v \leq nM$, let $f(i, v)$ be min size necessary to achieve profit exactly v using elements in $[i]$

If could compute this, could solve knapsack!

$$\text{OPT} = \max v \text{ s.t. } f(n, v) \leq K$$

Recurrence for $f(i, v)$:

$$f(i, v) = \begin{cases} 0 & i=0, v=0 \\ \infty & i=0, v>0 \\ f(i-1, v) & i>0, p(i)>v \\ \min(f(i-1, v), s(i) + f(i-1, v-p(i))) & i>0, p(i) \leq v \end{cases}$$

Running time:

time / table entry : $O(1)$

table entries : $n \cdot nM \sim n^2 M$

Pseudopolynomial !

Speed this up: "round" the instance to make M small

Let $\epsilon > 0$. We'll try for $(1-\epsilon)$ -approximation

$$\text{Let } \delta = \frac{\epsilon M}{n}$$

$$\text{Let } p'(i) = \left\lfloor \frac{p(i)}{\delta} \right\rfloor = \left\lfloor \frac{n}{\epsilon} \cdot \frac{p(i)}{M} \right\rfloor$$

"scales" so max profit $\frac{n}{\epsilon}$, "rounds" down to integer

Alg:

- Run old alg on p' profits!

Running time:

$$O(n^2 \frac{n}{\epsilon}) = O(\frac{n^3}{\epsilon})$$

Thm: $(1-\epsilon)$ -approximation

Pf: Let $I \subseteq [n]$ returned by algorithm,

$I^* \subseteq [n]$ optimal solution

Since did not round sizes, I feasible

$$p'(i) = \left\lfloor \frac{p(i)}{\delta} \right\rfloor$$

Def of $p'(i)$:

- $p(i) \geq \delta \cdot p'(i)$

- $p'(i) \geq \frac{p(i)}{\delta} - 1$

$$ALG = \sum_{i \in I} p(i)$$

$$\geq \delta \sum_{i \in I} p'(i)$$

$$\geq \delta \sum_{i \in I^*} p'(i)$$

(I is optimal for p')

$$\geq \delta \sum_{i \in I^*} \left(\frac{p(i)}{\delta} - 1 \right)$$

$$= \underbrace{\sum_{i \in I^*} p(i)}_{\text{OPT}} - \underbrace{\delta |I^*|}_{\leq \delta n}$$

$$\geq OPT - \delta n$$

$$(|I^*| \leq n)$$

$$\geq OPT - \epsilon M$$

$$(\delta = \frac{\epsilon M}{n})$$

$$\geq OPT - \epsilon \cdot OPT$$

$$(M \leq OPT)$$

$$= (1 - \epsilon) \cdot OPT$$

Fully Polynomial Time Approximation Scheme (FPTAS):

$(1 - \epsilon)$ -approximation, running time poly in $n, \frac{1}{\epsilon}$

Min-Makespan on Identical Parallel Machines

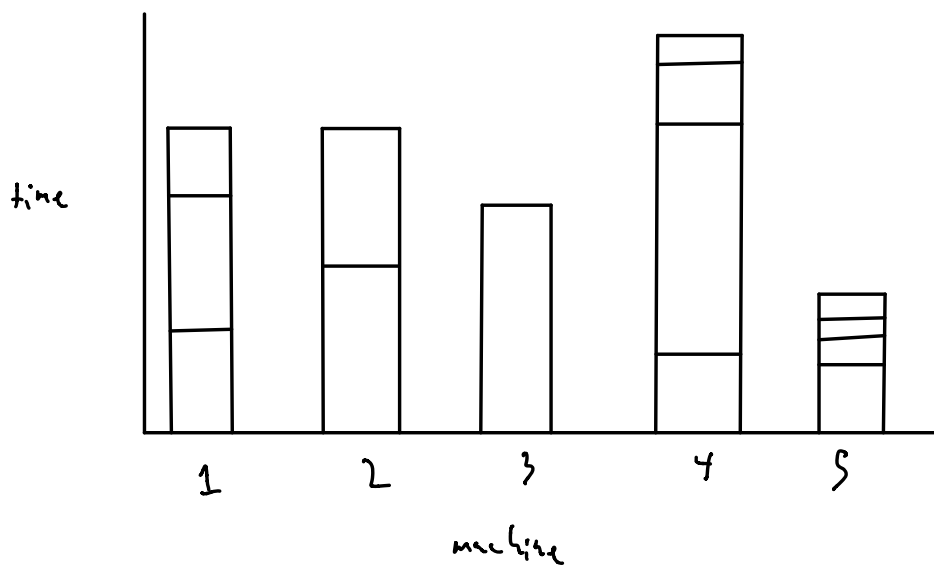
Input: Jobs J ($|J| = n$)

- Machines M ($|M| = k$)

- Processing times $p: J \rightarrow \mathbb{R}$

Feasible solution: $\mathbb{I}: J \rightarrow M$

Objective: min makespan = $\min_{m \in M} \max \sum_{j \in J: \mathbb{I}(j) = m} p(j)$



Greedy:

- Consider jobs in arbitrary order

- Assign job to machine with smallest load so far

Thm: Greedy is a 2-approximation

PF: Two easy observations:

$$1) p(j) \leq OPT \quad \forall j \in J$$

$$2) \frac{\sum_{j \in J} p(j)}{K} \leq OPT$$

Let $\mathbb{J}$ be greedy solution

Let m be machine most heavily loaded by greedy

Let j be last job assigned to m by greedy

Before j assigned:

$$\text{load on } m \leq \text{average load} \leq OPT$$

$$\Rightarrow \text{greedy} = \sum_{j' \in \mathbb{J}^{-1}(m)} p(j')$$

$$= \sum_{j' \in \mathbb{J}^{-1}(m) \setminus \{j\}} p(j') + p(j)$$

$$\leq OPT + OPT$$

$$= 2 \cdot OPT$$