

Local Search:

start with feasible solution, make "local" "improving" changes until get to local optimum.

Main difficulties:

- running time
- showing local opt are close to global

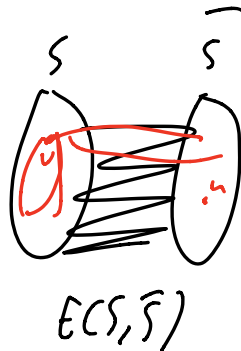
Max-Cut:

Input: $G=(V, E)$

Feasible: $S \subseteq V$

Objective: $\max |E(S, \bar{S})| = |\{ (u, v) \in E : u \in S, v \in \bar{S} \}|$

NP-hard, unlike Min-Cut



Alg: - Start with arbitrary $S \subseteq V$
- While $\exists v \in V$ s.t. v has more edges to same side of cut than other side, move v to other side

Thm: Running time is polynomial

PF:

Each iteration: poly time

iterations:

- Init: $|E(S, \bar{S})| \geq 0$

- $|E(S, \bar{S})|$ increases by ≥ 1 in every iteration

- $|E(S, \bar{S})| \leq m \leq n^2$

$\Rightarrow$ # iterations $\leq n^2$

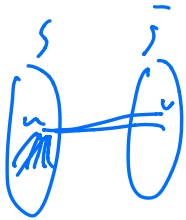
Thm: Local search is a 2-approximation

pf: A local optimum is an $S \subseteq V$ where switching any vertex doesn't help.

Alg returns a local opt.

Claim: every local opt is a 2-approx

Let S be local opt,



- $d(u)$ degree of u ,

- $d_{\text{across}}(u)$ #edges incident on u in $E(S, \bar{S})$

$$|E(S, \bar{S})| = \frac{1}{2} \sum_{u \in V} d_{\text{across}}(u)$$

$$\geq \frac{1}{2} \sum_{u \in V} \frac{1}{2} d(u) \quad (S \text{ local opt})$$

$$= \frac{1}{4} \sum_{u \in V} d(u)$$

$$= \frac{1}{4} \cdot 2M = \frac{M}{2}$$

$$\text{OPT} \leq M$$

$$\Rightarrow 2\text{-approx}$$

Weighted Max Cut:

Same as before, but also weights $w: E \rightarrow \mathbb{N}$

Goal: find $S \subseteq V$ maximizing $\sum_{e \in E(S, \bar{S})} w(e)$

$$\text{Let } W = \sum_{e \in E} w(e)$$

Previous algorithm:

Switch v to other side if it increases
weight across

Running time: - each iteration poly time
- $\leq W$ iterations

Input: n -ish bits for V $-\log W$ bits
 m -ish bits for E for each weight

Need to change algorithm!

Notation:

$$\delta(v) = \{e \in E : v \in e\}$$

$$w(E') = \sum_{e \in E'} w(e) \quad E' \subseteq E$$

$$\begin{aligned} w(S) &= w(E(S, \bar{S})) \quad S \subseteq V \\ &= \sum_{e \in E(S, \bar{S})} w(e) \end{aligned}$$

New Local Search Alg:

$$\text{Init: } S = \{v\}, \text{ where } v = \arg \max_{u \in V} w(\delta(u))$$

$$\text{while } \left(\begin{array}{l} \exists v \in S \text{ s.t. } w(S \setminus \{v\}) \geq (1 + \frac{\epsilon}{n}) w(S) \quad \text{or} \\ \exists v \in \bar{S} \text{ s.t. } w(S \cup \{v\}) \geq (1 + \frac{\epsilon}{n}) w(S) \end{array} \right)$$

change side of v

$$\underline{\text{Thm:}} \# \text{ iterations} \leq O\left(\frac{1}{\epsilon} n \log n\right)$$

PF: Let $S_0 = \{v\}$, $S_1, S_2, \dots, S_k$ be sets created by the algorithm.

$$\Rightarrow w(s_0) \geq \frac{w}{n}$$

$$\Rightarrow w(s_i) \geq (1 + \frac{\epsilon}{n}) w(s_{i-1}) \geq (1 + \frac{\epsilon}{n})^2 w(s_{i-2}) \geq \dots$$

$$\geq (1 + \frac{\epsilon}{n})^i w(s_0)$$

$$= (1 + \frac{\epsilon}{n})^i \frac{w}{n}$$

$$\Rightarrow \cancel{w(s_k)} \geq (1 + \frac{\epsilon}{n})^k \frac{w}{n}$$

$$\geq (1 + \frac{\epsilon}{n})^k \frac{\cancel{w(s_k)}}{n}$$

$$\Rightarrow (1 + \frac{\epsilon}{n})^k \leq n$$

$$\Rightarrow k \log(1 + \frac{\epsilon}{n}) \leq \log n$$

$$(1+x) \leq e^x \quad \forall x$$

$$\Rightarrow k \leq \frac{\log n}{\log(1 + \frac{\epsilon}{n})} \leq \frac{\log n}{\frac{\epsilon}{2n}}$$

$$\ln(1+x) \leq x$$

$$\log(1+x) \geq \frac{x}{2} \quad \text{for } 0 \leq x \leq 1$$

$$= O(\frac{1}{\epsilon} n \log n)$$



Thm: Alg is a $(2+\epsilon)$ -approx

Pf: Notation: $w(v, T) = \sum_{u \in T: \{v, u\} \in E} w(\{v, u\})$

Let S be set returned by algorithm.

Then $\forall v \in S$:

$$w(S \setminus \{v\}) \leq (1 + \frac{\epsilon}{n}) w(S) \quad (\text{def of alg})$$

$$\Rightarrow w(S \setminus \{v\}) - w(S) \leq \frac{\epsilon}{n} w(S)$$

$$\Rightarrow w(v, S) - w(v, \bar{S}) \leq \frac{\epsilon}{n} w(S)$$

$$\Rightarrow w(v, \bar{S}) \geq w(v, S) - \frac{\epsilon}{n} w(S)$$

$$\Rightarrow 2w(v, \bar{S}) \geq w(\delta(v)) - \frac{\epsilon}{n} w(S) \quad (\text{def } w(v, \bar{S}) \text{ for } v \in S)$$

$$\Rightarrow w(v, \bar{S}) \geq \frac{1}{2} w(\delta(v)) - \frac{\epsilon}{2n} w(S)$$

Symmetric argument:

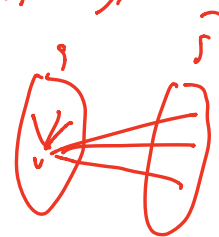
$$\forall v \in \bar{S}, \quad w(v, S) \geq \frac{1}{2} w(\delta(v)) - \frac{\epsilon}{2n} w(S)$$

$$\Rightarrow 2w(S) = \sum_{v \in S} w(v, \bar{S}) + \sum_{v \in \bar{S}} w(v, S)$$

$$\geq \sum_{v \in S} \left(\frac{1}{2} w(\delta(v)) - \frac{\epsilon}{2n} w(S) \right)$$

$$+ \sum_{v \in \bar{S}} \left(\frac{1}{2} w(\delta(v)) - \frac{\epsilon}{2n} w(S) \right)$$

$$= \sum_{v \in V} \left(\frac{1}{2} w(\delta(v)) - \frac{\epsilon}{2n} w(S) \right)$$



$$= W - \frac{\varepsilon}{2} u(s)$$

$$\Rightarrow (2 + \frac{\varepsilon}{2}) u(s) \geq W$$

$$\Rightarrow u(s) \geq \frac{W}{2 + \frac{\varepsilon}{2}} \geq \frac{OPT}{2 + \frac{\varepsilon}{2}}$$