Strengthening Relaxations: P-D for Min-Knapsack

Important technique/idea: strengthening relaxations

Min-Knapsack:

Input: - Items $I = [n]$ - Demand $D \in \mathbb{R}_{\geq 0}$

- values $v : I \to \mathbb{R}_{\geq 0}$ - sizes $s : I \to \mathbb{R}_{\geq 0}$

Feasible Solution: $X \subseteq I$ s.t. $\sum_{i \in X} v(i) \geq D$

Objective: min $s(X) = \sum_{i \in X} s(i)$

Note: WLOG $v(i) \leq D \quad \forall i \in I$

Obvious LP relaxation:

$$\min \sum_{i \in I} s(i) x_i$$

$$\text{s.t.} \sum_{i \in I} v(i) x_i \geq D$$

$$0 \leq x_i \leq 1 \quad \forall i \in I$$

Thm: ILP an exact formulation

Pf: Let x feasible integral solution to LP.

$\Rightarrow$ let $X = \{ i \in I : x_i = 1 \}$

$\Rightarrow \sum_{i \in X} v(i) = \sum_{i \in I} v(i) x_i \geq D$, so X feasible

$$s(X) = \sum_{i \in I} s(i) x_i$$

Let X feasible solution to Min-Knapsack.

$\Rightarrow$ let $x_i = \begin{cases} 1 & \text{if } i \in X \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \sum_{i \in I} v(i) x_i = \sum_{i \in X} v(i) \geq D \qquad (X \text{ feasible})$$

$$\sum_{i \in I} s(i) x_i = \sum_{i \in X} s(i)$$

So ILP exact. Is LP relaxation good?

<u>Ex</u>: $I = \{1, 2\}$

$v(1) = D - 1 \qquad s(1) = 0$

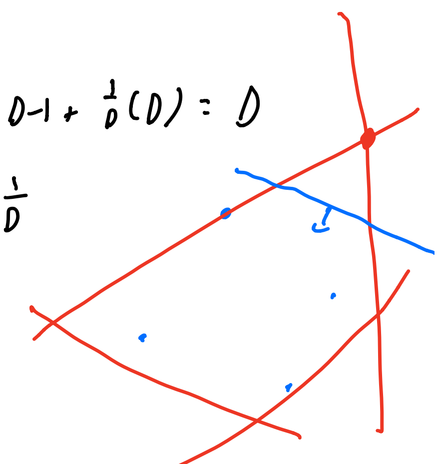
$v(2) = D \qquad s(2) = 1$

Only feasible solution: take item 2 $\Rightarrow$ $OPT = \underline{1}$

LP solution:

$x_1 = 1 \qquad x_2 = \frac{1}{D} \qquad \Rightarrow \quad \sum_{i=1}^{2} v(i) x_i = D-1 + \frac{1}{D}(D) = D$

$$\sum_{i=1}^{2} s(i) x_i = \frac{1}{D}$$



$\Rightarrow$ integrality gap: $\frac{OPT}{LP} = \frac{1}{1/D} = D$

Can be exponential in size of input!

Can we add valid constraints (all integral solutions still feasible) to make LP better?

Different from Multiway Cut: adding extra constraints, not completely new LP

What went wrong in example?

- Bought "partial solution" of item 1

  $\Rightarrow$ "remaining demand" = 1,

  value of item 2 = $D > 1$

  First WLOG was $v(i) \le D \ \forall i$

- After buying item 1, should "pretend" $v(2) = 1$, not $v(2) = D$

Generalize: suppose promise to buy $A \subseteq I$.

Constraint for "remaining" problem:

$$\sum_{i \in I} v(i) x_i \ge D$$

$$\sum_{i \in I \setminus A} \min(v(i), D - v(A)) x_i \ge D - v(A)$$

New LP: include all of these constraints!

$$\min \sum_{i \in I} s(i) x_i$$

$$= \sum_{i \in A} v(i)$$

$$\text{s.t.} \sum_{i \in I \setminus A} \min(v(i), D - v(A)) x_i \geq D - v(A) \qquad \forall A \subseteq I$$

$$x_i \geq 0 \qquad \forall i \in I$$

Note: when $A = \emptyset$ have original constraint

Knapsack Cover Inequalities: Carr, Fleischer, Leung, Phillips '00

why is this valid?

Lemma: Let X be feasible solution to Min-Knapsack.

Then $x_i = \begin{cases} 1 & \text{if } i \in X \\ 0 & \text{otherwise} \end{cases}$ is feasible LP solution

Pf: Consider some $A \subseteq I$.

$$X \text{ feasible} \Rightarrow \sum_{i \in X} v(i) \geq D$$

$$\Rightarrow \sum_{i \in X \setminus A} v(i) + \sum_{i \in X \cap A} v(i) \geq D$$

$$\Rightarrow \sum_{i \in X \setminus A} v(i) \geq D - v(A \cap X)$$

$$\geq D - v(A)$$

$$\Rightarrow \sum_{i \in X \setminus A} \min(v(i), D - v(A)) \geq D - v(A)$$

(if $\exists i$ s.t. $v(i) \geq D - v(A)$ satisfied, o/w also satisfied)

$$\Rightarrow \sum_{i \in X \setminus A} \min(v(i), D - v(A)) x_i \geq D - v(A) \quad \text{(def of } x_i\text{)}$$

$$\Rightarrow \sum_{i \in I \setminus A} \min(v(i), D - v(A)) x_i \geq D - v(A)$$

Sanity check: what does this do to our example?

$$v(1) = D - 1 \qquad s(1) = 0$$
$$v(2) = D \qquad s(2) = 1$$

Consider $A = \{1\}$

$\Rightarrow$ LP constraint for A: $\min(v(2), D - v(A)) x_2 \geq D - v(A)$

$$\Rightarrow \min(D, 1) x_2 \geq 1$$

$$\Rightarrow x_2 \geq 1$$

Primal-Dual Algorithm:

Primal:

$$\min \sum_{i \in I} s(i) x_i$$

$$\text{s.t.} \sum_{i \in I \setminus A} \min(v(i), D - v(A)) x_i \geq D - v(A) \qquad \forall A \subseteq I$$

$$x_i \geq 0 \qquad \forall i \in I$$

Dual: variable y_A $\forall A \subseteq I$, constraint $\forall i \in I$

$$\max \sum_{A \subseteq I} (D - v(A)) \, y_A$$

$$\text{s.t.} \quad \sum_{A \subseteq I : i \notin A} \min(v(i), D - v(A)) \, y_A \leq s(i) \qquad \forall i \in I$$

$$0 \leq y_A \qquad \forall A \subseteq I$$

Alg:

- Init $y = \vec{0}$, $X = \emptyset$
- while $v(X) < D$ {
  - increase y_X until some dual constraint becomes tight:

    $i \in I \setminus X$ s.t. $\displaystyle\sum_{A \subseteq I : i \notin A} \min(v(i), D - v(A)) \, y_A = s(i)$
  - $X = X \cup \{i\}$

  }
- Return X

Lemma: $\vec{y}$ is feasible dual solution throughout algorithm

Pf: True initially: $y = \vec{0}$

Consider some iteration, y feasible at beginning

$\Rightarrow$ X set at beginning of iteration.

If y_X increases, all i s.t. dual constraint tight are in X

$\Rightarrow$ still feasible at end of iteration

<u>Thm</u>: 2-approximation

<u>Pf</u>: X set returned by alg.

$$s(X) = \sum_{i \in X} s(i)$$

$$= \sum_{i \in X} \sum_{A \subseteq I : i \notin A} \min(v(i), D - v(A))\, y_A \qquad \text{(def of alg)}$$

$$= \sum_{A \subseteq I} \sum_{i \in X \setminus A} \min(v(i), D - v(A))\, y_A \qquad \text{(order of summation)}$$

$$= \sum_{A \subseteq I} y_A \underbrace{\sum_{i \in X \setminus A} \min(v(i), D - v(A))}_{\text{want to bound for } A \text{ s.t. } y_A > 0}$$

Let $A : y_A > 0$, $\Rightarrow$ A raised by alg $\Rightarrow A \subseteq X$

Let ℓ be last item added by alg, $\Rightarrow \ell \in X \setminus A$

Let $i \in X \setminus A$, $i \neq \ell$

$\Rightarrow v(A \cup \{i\}) < D$, since o/w alg would have finished before adding ℓ

$\Rightarrow v(A) + v(i) < D$

$\Rightarrow v(i) < D - v(A)$

$\Rightarrow \min(v(i), D - v(A)) = v(i)$

$$= \sum_{A \subseteq I} y_A \left(\min(v(\ell), D - v(A)) + \sum_{i \in X \setminus A : i \neq \ell} v(i) \right)$$

$$= \sum_{A \subseteq I} y_A \left(\min(v(\ell), D - v(A)) + v(X \setminus \{\ell\}) - v(A) \right)$$

$$< \sum_{A \subseteq I} y_A \left(D - v(A) + D - v(A) \right)$$

$$= 2 \sum_{A \subseteq I} (D - v(A))\, y_A$$

$$\leq 2 \cdot OPT$$ (weak duality)