LP Duality:

Given an LP and a feasible solution, how could I prove to you that solution is optimal?

Ex:

$$\begin{aligned}
\min \quad & 3x_1 + x_2 + 4x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 \geq 3 && \text{C1} \\
& x_1 + 2x_3 \geq 2 && \text{C2} \\
& 2x_1 + 3x_2 + x_3 \geq 4 && \text{C3} \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

Consider solution $(0, \frac{3}{2}, 1)$

$\Rightarrow$ value $= \frac{3}{2} + 4 \cdot 1 = \frac{11}{2}$.

Is this optimal?

Idea: Combine constraints to get new constraints.

- Any solution satisfying C1, C2, C3 will satisfy any nonnegative linear combination of them!

Ex: $\frac{1}{2}(\text{C2}) + \frac{1}{3}(\text{C3})$:

$$\frac{1}{2}(x_1 + 2x_3) + \frac{1}{3}(2x_1 + 3x_2 + x_3) \geq \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 4$$

$\Leftrightarrow \left(\frac{1}{2}+\frac{2}{3}\right)x_1 + x_2 + \left(1+\frac{1}{3}\right)x_3 \geq 7/3$

$\Leftrightarrow \frac{7}{6}x_1 + x_2 + \frac{4}{3}x_3 \geq 7/3$

Note: $7/6 \leq 3$, $1 \leq 1$, and $4/3 \leq 4$.

And all variables nonnegative $3x_1 + x_2 + 4x_3$

$\Rightarrow$ for any LP solution x,

$$3x_1 + x_2 + 4x_3 \geq \frac{7}{6}x_1 + x_2 + 4/3\, x_3 \geq 7/3$$

So proved LP OPT $\geq 7/3$!

What's the best lower bound we can prove using this method?

Multipliers $\overset{0}{y_1}$ of (1, $\overset{\frac{1}{2}}{y_2}$ of (2, $\overset{\frac{1}{3}}{y_3}$ of (3

Need: $y_1, y_2, y_3 \geq 0$

$y_1 + y_2 + 2y_3 \leq 3$ (coefficient of x_1 in comb. ≤ 3)

$2y_1 + 3y_3 \leq 1$ (coefficient of x_2 in comb. ≤ 1)

$2y_2 + y_3 \leq 4$ (coefficient of x_3 in comb. ≤ 4)

want to max lower bound we get:

$$\max \ 3y_1 + 2y_2 + 4y_3$$

This is an LP! Called the dual LP

Solve it: get $y_1 = \frac{1}{2}$, $y_2 = 2$, $y_3 = 0$

$\Rightarrow$ objective value $\frac{3}{2} + 4 = 11/2$

$\Rightarrow$ original (primal) LP has LP OPT $\geq 11/2$

So original solution was optimal!

More general:

LP with n variables, m constraints

$c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$

$$\begin{aligned} \min \ & c^T x \\ \text{s.t. } & Ax \geq b \\ & x \geq 0 \end{aligned} \quad = \quad \begin{aligned} \min \ & \sum_{j=1}^{n} c_j x_j \\ \text{s.t. } & \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \forall i \in [m] \\ & x_j \geq 0 \quad \forall j \in [n] \end{aligned}$$

Dual: var y_i $\forall i \in [m]$, constraint for each $j \in [n]$

$$\begin{aligned} \max\ & b^T y \\ \text{s.t. } & A^T y \le c \\ & y \ge 0 \end{aligned} \quad = \quad \begin{aligned} \max\ & \sum_{i=1}^{m} b_i y_i \\ \text{s.t. } & \sum_{i=1}^{m} a_{ij} y_i \le c_j \quad && \forall j \in [n] \\ & y_i \ge 0 && \forall i \in [m] \end{aligned}$$

<u>Note</u>: Can put any LP into either form, since didn't assume A, b, c nonnegative:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \iff \sum_{j=1}^{n} (-a_{ij}) x_j \ge -b_i$$

<u>Thm</u>: If D is dual of P, then P is dual of D

<u>Thm</u> (Weak Duality): For any feasible primal-dual solution pair (x, y), $b^T y \le c^T x$ (OPT(dual) ≤ OPT(primal))

<u>Pf</u>:

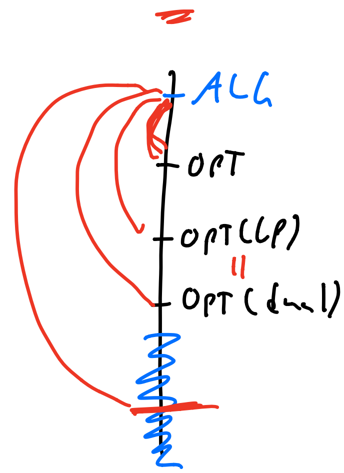
$$c^T x = \sum_{j=1}^{n} c_j x_j \ge \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j \qquad \text{(dual constraints)}$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i$$

$$\ge \sum_{i=1}^{m} b_i y_i = b^T y \qquad \text{(primal constraints)}$$

Already very useful!

(integral) Minimization problem:



Thm (Strong Duality): If x^* optimal for primal, y^* optimal for dual, then $b^T y^* = c^T x^*$.

OPT(dual) = OPT(primal)

(If primal infeasible then dual unbounded, if dual infeasible then primal unbounded)

Not going to prove today

Thm (Complementary Slackness): Let (x, y) feasible (primal, dual) solutions. Then (x, y) both optimal iff:

1) $\forall j \in [n]$: $\sum_{i=1}^{m} a_{ij} y_i = c_j$ or $x_j = 0$ (or both)

2) $\forall i \in [m]$: $\sum_{j=1}^{n} a_{ij} x_j = b_i$ or $y_i = 0$ (or both)

How to think about this: if a variable is nonzero (by any amount), then dual constraint is tight.

<u>Pf</u>: Sps CS conditions hold

$\Rightarrow$ Look at proof of weak duality

$\Rightarrow$ all inequalities tight!

$\Rightarrow c^T x = b^T y$

$\Rightarrow$ both optimal (weak duality)

Sps (x,y) optimal

$\Rightarrow c^T x = b^T y$ (strong duality)

$\Rightarrow$ all inequalities tight in weak duality proof

$\Rightarrow$ CS conditions hold

Examples:

Min s-t cut: $P = \{\text{all s-t paths}\}$

$$\min \sum_{e \in E} c(e)\, x_e$$

$$\text{s.t.} \quad \sum_{e \in P} 1\, x_e \geq 1 \qquad \forall P \in \mathcal{P}$$

$$x_e \geq 0 \qquad \forall e \in E$$

Know can round without losing anything: there is integral x s.t. x feasible, $c^T x$ optimal LP solution

Dual: variable y_P for each $P \in \mathcal{P}$

constraint for every $e \in E$

$$\max \sum_{P \in \mathcal{P}} y_P$$

$$\text{s.t.} \quad \sum_{P \in \mathcal{P}: e \in P} 1\, y_P \leq c(e) \qquad \forall e \in E$$

$$y_P \geq 0 \qquad \forall P \in \mathcal{P}$$

Max-flow LP!

Let y optimal flow $\Rightarrow c^T x = 1^T y$ (strong duality)

$\Rightarrow$ max flow = min cut!

Multicut: $P_i = \{s_i - t_i \text{ paths}\}$ for each $i \in [k]$

$$\min \sum_{e \in E} c(e) x_e$$

$$\text{s.t.} \sum_{e \in P} x_e \geq 1 \qquad \forall i \in [k], \forall P \in P_i$$

$$x_e \geq 0 \qquad \forall e \in E$$

Dual: Variable y_P for each $i \in [k]$, $P \in P_i$

Constraint for each $e \in E$

$$\max \sum_{i=1}^{k} \sum_{P \in P_i} y_P$$

$$\text{s.t.} \sum_{i=1}^{k} \sum_{P \in P_i : e \in P} y_P \leq c(e) \qquad \forall e \in E$$

$$y_P \geq 0 \qquad \forall i \in [k], \forall P \in P_i$$

Max multicommodity flow!

F^* value of max multicommodity flow

c^* val of min fractional multicut

$\hat{c}$ val of min multicut

Thm: $F^* \le \hat{C} \le 4\ln(k+1) \cdot F^*$

Pf:

$$F^* \underset{\text{weak duality}}{\le} C^* \underset{\text{relaxation}}{\le} \hat{C} \underset{\text{rounding}}{\le} 4\ln(k+1)\, C^* \underset{\text{strong duality}}{=} 4\ln(k+1) \cdot F^*$$

Flow-Cut gap