Multicut:

Input: - $G=(V,E)$

- $c: E \to \mathbb{R}^+$

- k pairs $(s_1,t_1),\dots,(s_k,t_k)$

Feasible solution: $F \subseteq E$ s.t. $\forall i \in [k]$, no $s_i - t_i$ path in $G \setminus F$

Objective: min $c(F) = \sum_{e \in F} c(e)$

Last time: $O(\log n)$-approx

Today: $O(\log k)$-approx

$P_i = \{ s_i - t_i \text{ paths} \}$

LP:

$$\min \sum_{e \in E} c(e) x_e$$

$$\text{s.t.} \sum_{e \in P} x_e \geq 1 \qquad \forall i \in [k], \forall P \in P_i$$

$$0 \leq x_e \leq 1 \qquad \forall e \in E$$

Solving LP: Ellipsoid, separation oracle = shortest path

Let x optimal LP solution

$$V^* = \sum_{e \in E} c(e) x_e = LP$$

$d : V \times V \to \mathbb{R}_{\geq 0}$ shortest path metric using x as edge lengths

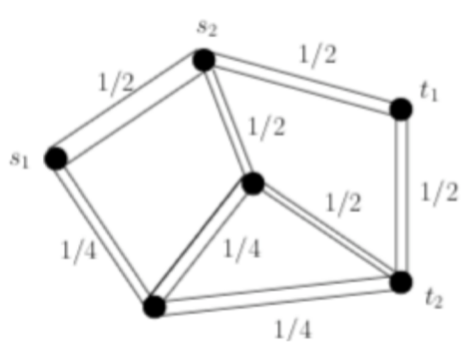
$\Rightarrow d(s_i, t_i) \geq 1$ by LP (but no bound on $d(s_i, s_j)$)

$\delta(S) = E(S, \bar{S})$

Metaphor: x_e = length of edge e

Think of $c(e)$ as "cross-section area" of e

$\Rightarrow c(e) x_e$ = "volume" of e.



V^* = total volume of G

Def: (Normalized) Volume:

$$V(s_i, r) = \frac{V^*}{k} + \sum_{\substack{e \in E: \\ e \subseteq B(s_i, r)}} c(e) x_e + \sum_{\substack{e = \{u,v\} \in E: \\ u \in B(s_i, r) \\ v \notin B(s_i, r)}} c(e)(r - d(s_i, u))$$

useful for later calculation

edges inside ball

edges cross ball

Lemma (Region Growing): For all $i \in [k]$, in poly time we can find some $0 \le r < \frac{1}{2}$ s.t.

$$c(\delta(B(s_i, r))) \le 2 \ln(k+1) \cdot V(s_i, r)$$

Prove lemma later. Assuming it, rounding alg:

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F = ∅
for i = 1 to k {
    if (s_i, t_i connected in G\F) {
        Let r_i be radius from lemma for i
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$F = F \cup \delta(B(s_i, r_i))$

Remove $B(s_i, r_i)$ and incident edges from graph

}

}

Note: Lemma applied to <u>changing</u> graph

<u>Thm</u>: Returns feasible solution

<u>Pf</u>: $\forall i$, component of $G \setminus F$ containing s_i is $B(s_i, r_i)$

for some $r_i < \frac{1}{2}$.

But $d(s_i, t_i) \geq 1$

$\Rightarrow s_i, t_i$ not connected in $G \setminus F$

<u>Thm</u>: $c(F) \leq 4 \ln(k+1) \cdot V^* \leq 4 \ln(k+1) \cdot OPT$

<u>Pf</u>: Definitions:

- $B_i = B(s_i, r_i)$ in iteration i

(If ball not constructed in iteration i b/c s_i, t_i already disconnected, $B_i = \emptyset$)

(Ball constructed in iteration i; not necessarily $B(s_i, r_i)$ in original graph)

- F_i := edges added to cut in iteration i = $\delta(B(s_i, r_i)) = \delta(B_i)$

in iteration i graph

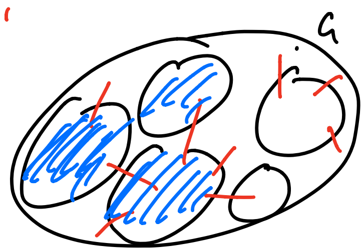
$\Rightarrow F = \bigcup_{i=1}^{k} F_i, \quad F_i \cap F_j = \emptyset \quad \forall i \neq j \in [k]$

- V_i := volume of edges removed in iteration i

$$= \sum_{\substack{e=\{u,v\}: \\ u,v \in B_i}} c(e)\,x_e + \sum_{e \in F_i} c(e)\,x_e$$

$\Rightarrow V_i \geq V(s_i, r_i) - \frac{V^*}{k}$, since V_i includes full volume of cut edges but not normalization term

$\sum_{i=1}^{k} V_i \leq V^*$, since every edge contributes to V_i for at most one i



(region growing)

$$\Rightarrow c(F) = \sum_{i=1}^{k} c(F_i)$$

$$\leq 2\ln(k+1) \sum_{i=1}^{k} V(s_i, r_i)$$

$$\leq 2\ln(k+1) \sum_{i=1}^{k} \left(V_i + \frac{V^*}{k}\right)$$

$$\leq 2\ln(k+1)\left(V^* + V^*\right) = 4\ln(k+1) \cdot V^*$$

So just need to prove region growing lemma

<u>Lemma (Region Growing)</u>: For all $i \in [k]$, in poly time we can find some $0 \le r < \frac{1}{2}$ s.t.

$$c(\delta(B(s_i, r))) \le 2\ln(k+1) \cdot V(s_i, r)$$

<u>Pf</u>: Simplify notation:

$c(r) = c(\delta(B(s_i, r)))$ $\quad c: \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$

$V(r) = V(s_i, r)$ $\quad V: \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$

$\Rightarrow$ WTS can find $r < \frac{1}{2}$ s.t. $c(r) \le 2\ln(k+1) \cdot V(r)$

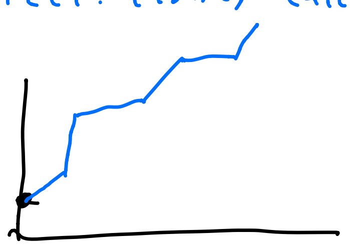
$\Rightarrow \frac{c(r)}{V(r)} \le 2\ln(k+1)$

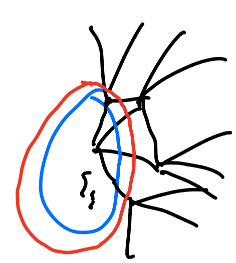
<u>Algorithm</u> (for now)

Choose r uniformly at random in $(0, 1/2)$

WTS: $E\left[\frac{c(r)}{V(r)}\right] \le 2\ln(k+1)$

Main idea: (abuse) calculus





$$V(r) = \frac{V^*}{k} + \sum_{\substack{e \in E: \\ e \subseteq B(s_i, r)}} c(e) x_e + \sum_{\substack{e = \{u,v\} \in E: \\ u \in B(s_i, r) \\ v \notin B(s_i, r)}} c(e)(r - d(s_i, u))$$

Order $B(s_i, \frac{1}{2})$: $s_i = v_0, v_1, v_2, \ldots, v_m$ s.t. $d(s_i, v_i) \leq d(s_i, v_{i+1})$

Let $r_j = d(s_i, v_j)$

If $r \in (r_j, r_{j+1})$:

$$\frac{dV(r)}{dr} =$$

$$\sum_{\substack{e = \{u,v\} \in E: \\ u \in B(s_i, r) \\ v \notin B(s_i, r)}} c(e)$$

$$= c(r)$$

If $V(r)$ continuous, differentiable (it's not):

$$E\left[\frac{c(r)}{V(r)}\right] = \frac{1}{\frac{1}{2}} \int_0^{1/2} \frac{c(r)}{V(r)} \, dr$$

$$= 2 \int_0^{\frac{1}{2}} \frac{1}{V(r)} \frac{dV(r)}{dr} \, dr$$

$$= 2\int_0^{\frac{1}{2}} \frac{1}{V(r)}\, dV(r)$$ $x = V(r) \Rightarrow \int_0^{\frac{1}{2}} \frac{1}{x}\, dx$

$$= 2\left(\ln\left(V\left(\tfrac{1}{2}\right)\right) - \ln(V(0))\right)$$

$$= 2\ln\left(\frac{V(\frac{1}{2})}{V(0)}\right)$$ $V(0) \geq \frac{v^*}{k}$ $\quad V(\frac{1}{2}) \leq V^* + \frac{v^*}{k}$

$$\leq 2\ln\left(\frac{\frac{v^*}{k} + V^*}{\frac{v^*}{k}}\right)$$

$$= 2\ln(k+1)$$

$\Rightarrow$ by mean value thm, $\exists r$ s.t. $\frac{c(r)}{V(r)} \leq 2\ln(k+1)$

Not continuous or differentiable! See notes/book

Breakpoints at r_j's, linear increasing b/w breaks.

Actually finding r:

$r \in [r_j, r_{j+1})$: $c(r)$ same for all r in interval

$V(r)$ maximized when just less than r_{j+1}

$\Rightarrow \frac{c(r)}{V(r)}$ minimized in $[r_j, r_{j+1})$ just below r_{j+1}

$\Rightarrow$ best choice of r some $r_i - dr$

Try each one, take best.