

Multway cut:

Input: - $G = (V, E)$ (undirected)

- costs $c: E \rightarrow \mathbb{R}^+$

- $T = \{s_1, s_2, \dots, s_k\} \subseteq V$

Feasible: $A \subseteq E$ s.t. $G \setminus A$ has no $s_i - s_j$ path
 $\forall i \neq j \in [k]$

Objective: $\min \sum_{e \in E} c(e)$

Last class: 2-approximation, $2(1 - \frac{1}{k})$ integrality gap.

Trying to do better via new LP

Think of opt cut $F \subseteq E$.

Let $C_i = \{u \in V: u \text{ reachable from } s_i \text{ in } G \setminus F\}$

$\Rightarrow C_i \cap C_j = \emptyset \quad \forall i \neq j \in [k]$

Claim: C_i 's form partition

New LP:

Variables: $x_u^i = \begin{cases} 1 & \text{if } u \in C_i \\ 0 & \text{otherwise} \end{cases}$

$$z_e^i = \begin{cases} 1 & \text{if } e \in \delta(C_i) = E(C_i, \bar{C}_i) \\ 0 & \text{otherwise} \end{cases}$$

$$\min \quad \frac{1}{2} \sum_{e \in E} \sum_{i=1}^k c(e) z_e^i \quad \sum_{e \in E} c(e) \left(\frac{1}{2} \sum_{i=1}^k z_e^i \right)$$

$$\text{s.t.} \quad \sum_{i=1}^k x_u^i = 1 \quad \forall u \in V$$

$$x_{s_i}^i = 1 \quad \forall i \in [k]$$

$$z_e^i \geq x_u^i - x_v^i \quad \forall e = \{u, v\} \in E, \forall i \in [k]$$

$$z_e^i \geq x_v^i - x_u^i \quad \forall e = \{u, v\} \in E, \forall i \in [k]$$

$$0 \leq x_u^i \leq 1 \quad \forall u \in V, \forall i \in [k]$$

$$0 \leq z_e^i \leq 1 \quad \forall e \in E, \forall i \in [k]$$

Thm: ILP version is exact formulation

Pf: Exercise

Rewrite in more compact way

$$\|x-y\|_2 = \sqrt{\sum_{i=1}^k (x^i - y^i)^2}$$

Def: For $x, y \in \mathbb{R}^k$, $\|x-y\|_1 = \sum_{i=1}^k |x^i - y^i|$,

where x^i is i 'th coordinate of x
 y^i is i 'th coordinate of y

Def: k -simplex $\Delta_k = \{x \in \mathbb{R}^k : \sum_{i=1}^k x^i = 1 \text{ and } x^i \geq 0 \forall i\}$

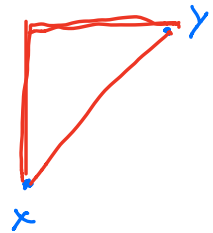
Def: e_i is vector with 1 in coordinate i ,
0 in all other coordinates

Given LP solution (x, z) , let $x_u = (x_u^1, x_u^2, \dots, x_u^k)$

In any optimal LP solution, $z_e^i = |x_u^i - x_v^i|$

$$\Rightarrow \|x_u - x_v\|_1 = \sum_{i=1}^k z_e^i \quad e = \{u, v\}$$

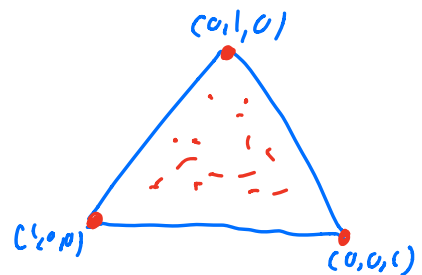
Rewritten LP:



$$\min \frac{1}{2} \sum_{e \in E} c(e) \|x_u - x_v\|_1 = \frac{1}{2} \sum_{e \in E} c(e) \sum_{i=1}^k 2x_e^i$$

$$\text{s.t. } x_{s_i} = e_i \quad \forall i \in [k]$$

$$x_u = \Delta_k$$



Not technically LP, but equivalent to linear LP

In other words: embed vertices into k -simplex,
terminals at corners, min weighted ℓ_1 -distances!

$\Rightarrow$ integral solutions: all nodes at corners

$$\text{Note: } \|x_{s_i} - x_{s_j}\|_1 = 2$$

$$\text{Def: } B(s_i, r) = \{v \in V : \|e_i - x_v\|_1 \leq r\}$$

Rounding Algorithm [Calinescu, Karloff, Rabani]:

- Initially $C_i = \emptyset \quad \forall i \in [k]$

$$X = \emptyset$$

- Pick $r \in (0, 2)$ uniformly at random

- Pick π a permutation of $[k]$ u.a.r.

- for $i = 1$ to $k-1$

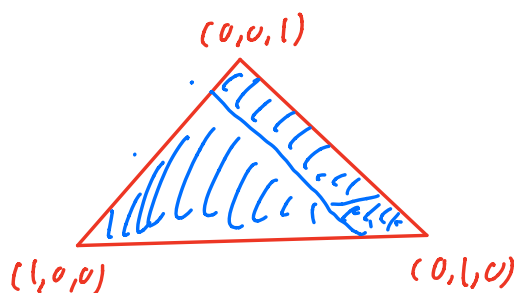
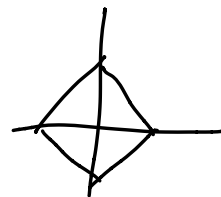
$$- C_{\pi(i)} = B(s_{\pi(i)}, r) \setminus X$$

$$- X = X \cup C_{\pi(i)}$$

- Set $C_{\pi(k)} = V \setminus X$

- Return $A = \bigcup_{i=1}^k \delta(C_i)$

Basically "one level" of FRT!



$\pi = (1, 3, 2)$

Thm: kR is a $\frac{3}{2}$ -approximation

Random variable: $Z_{u,v} = \begin{cases} 1 & \text{if } u,v \text{ separated by cut} \\ 0 & \text{otherwise} \end{cases}$

Claim: $\Pr[Z_{u,v} = 1] \leq \frac{3}{4} \|x_u - x_v\|_1 \quad \forall \{u,v\} \in E$

Assuming claim:

$$\begin{aligned} E[ALG] &= E\left[\sum_{e=\{u,v\} \in E} c(e) Z_{u,v}\right] \\ &= \sum_{e=\{u,v\} \in E} c(e) E[Z_{u,v}] \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{e=\{u,v\} \in E} c(e) \frac{3}{4} \|x_u - x_v\|_1 \\
&= \frac{3}{2} \cdot \frac{1}{2} \cdot \sum_{e=\{u,v\} \in E} c(e) \|x_u - x_v\|_1 \\
&= \frac{3}{2} \cdot LP \leq \frac{3}{2} \cdot OPT
\end{aligned}$$

Pf of claim:

$$Pr[Z_{uv} = 1] \leq \frac{3}{4} \|x_u - x_v\|_1 \quad \forall \{u,v\} \in E$$

Start like FRT. Fix $e=\{u,v\} \in E$.

$$s_i = \begin{cases} 1 & \text{if } i \text{ first index in } \pi \text{ s.t. } B(s_i, r) \cap \{u,v\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

s_i settles u,v

$$X_i = \begin{cases} 1 & \text{if } |B(s_i, r) \cap \{u,v\}| = 1 \\ 0 & \text{otherwise} \end{cases}$$

s_i cuts u,v

Observation: $Z_{uv} = \sum_{i=1}^k s_i X_i$

(one i settles, separated if it also cuts)

$$\begin{aligned}
\Rightarrow P_r[Z_w = 1] &= E[Z_w] = E\left[\sum_{i=1}^k S_i X_i\right] = \sum_{i=1}^k E[S_i X_i] \\
&= \sum_{i=1}^k P_r[S_i = 1 \wedge X_i = 1] \\
&= \sum_{i=1}^k P_r[S_i = 1 | X_i = 1] \cdot P_r[X_i = 1]
\end{aligned}$$

useful Lemma: relate distances to single coordinate

Lemma: $\|e_i - x_w\|_1 = 2(1 - x_w^i) \quad \forall w \in V, i \in [k]$

Pf: $\|e_i - x_w\|_1 = \sum_{j=1}^k |e_i^j - x_w^j|$

$$\begin{aligned}
&= 1 - x_w^i + \sum_{j \neq i} x_w^j \\
&= 1 - x_w^i + (1 - x_w^i) \quad \left(\sum_{j=1}^k x_w^j = 1\right) \\
&= 2(1 - x_w^i)
\end{aligned}$$

w.l.o.g., $x_w^i \geq x_w^j$

So $P_r[X_i = 1] = P_r[|B(i, r) \cap \{u, v\}| = 1]$

$$= P_r[\min(\|e_i - x_u\|_1, \|e_i - x_v\|_1) \leq r < \max(\|e_i - x_u\|_1, \|e_i - x_v\|_1)]$$

$$= P_r[\min(2(1 - x_u^i), 2(1 - x_v^i)) \leq r < \max(2(1 - x_u^i), 2(1 - x_v^i))]$$

$$= \Pr[2(1-x_u^i) \leq r < 2(1-x_v^i)]$$

$$= \frac{2(1-x_v^i) - 2(1-x_u^i)}{2}$$

$\swarrow$ r uniformly distributed in $(0, 2)$

$$= x_u^i - x_v^i$$

$$= |x_u^i - x_v^i|$$

$$\text{Let } l = \underset{i \in [k]}{\operatorname{argmin}} \min(1-x_u^i, 1-x_v^i)$$

s_l terminal closest to $\{u, v\}$ in l_1 -metric

$$\Rightarrow \Pr[s_l = 1 \mid X_l = 1] \cdot \Pr[X_l = 1] \leq \Pr[X_l = 1]$$

$$\approx |x_u^l - x_v^l|$$

consider some $i \neq l$

$$\text{If } X_i = 1 \quad (i \text{ cuts } u, v \Rightarrow B(s_l, r) \cap \{u, v\} \neq \emptyset)$$

$$\Rightarrow s_i = 1 \text{ only if } i \text{ before } l \text{ in } \pi$$

$$\Rightarrow \Pr[s_i = 1 \mid X_i = 1] \leq \frac{1}{2}$$

$$\Rightarrow P(S_i = 1 | X_i = 1) \cdot P(X_i = 1) \leq \frac{1}{2} |x_u^i - x_v^i|$$

Finish up:

$$P(Z_{uv} = 1) = \sum_{i=1}^k P(S_i = 1 | X_i = 1) \cdot P(X_i = 1)$$

$$\leq |x_u^l - x_v^l| + \sum_{i \neq l} \frac{1}{2} |x_u^i - x_v^i|$$

$$= \frac{1}{2} \underbrace{|x_u^l - x_v^l|}_{\leq \frac{1}{2} \|x_u - x_v\|_1} + \frac{1}{2} \|x_u - x_v\|_1$$

$$\leq \frac{3}{4} \|x_u - x_v\|_1$$

Lemma: $|x_u^l - x_v^l| \leq \frac{1}{2} \|x_u - x_v\|_1$

pf: $u, v \in L \cup R, x_u^l \geq x_v^l$

$$\Rightarrow |x_u^l - x_v^l| = x_u^l - x_v^l = \left(1 - \sum_{i \neq l} x_u^i\right) - \left(1 - \sum_{i \neq l} x_v^i\right)$$

$$= \sum_{i \neq l} (x_v^i - x_u^i)$$

$$\leq \sum_{i \neq l} |x_v^i - x_u^i|$$

$$\Rightarrow 2|x_u^l - x_v^l| \leq \|x_u - x_v\|_1$$

$$\Rightarrow |x_u^l - x_v^l| \leq \frac{1}{2} \|x_u - x_v\|_1$$

Multicut:

Input: - $G = (V, E)$

- $c: E \rightarrow \mathbb{R}^+$

- Pairs $(s_1, t_1), (s_2, t_2), \dots, (s_k, t_k)$

Feasible solution: $F \subseteq E$ s.t. $\forall i \in [k]$, no $s_i - t_i$ path
in $G \setminus F$

Objective: $\min c(F) = \sum_{e \in F} c(e)$

Thm: There is an $O(\log n)$ -approximation for Multicut

Def: $P_i = \{s_i - t_i \text{ paths}\}$

LP:

$$\min \sum_{e \in E} c(e) x_e$$

$$\text{s.t.} \quad \sum_{e \in P} x_e \geq 1 \quad \forall i \in [k], \forall P \in P_i$$

$$0 \leq x_e \leq 1 \quad \forall e \in E$$

Solving LP: Ellipsoid, separation oracle = shortest path

Let x optimal LP solution

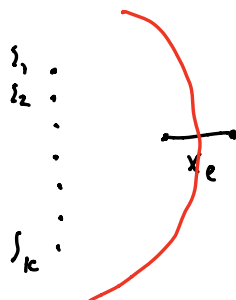
$$V^* = \sum_{e \in E} c(e) x_e = LP$$

$d: V \times V \rightarrow \mathbb{R}_{\geq 0}$ shortest path metric using x
as edge lengths

$\Rightarrow d(s_i, t_i) \geq 1$ by LP (but no bound on $d(s_i, s_j)$)

How to round?

Ball around each s_i (like Multicommodity (cut));



Still want to use random-radius balls, but how?

Def: Given metric space (V, d) , a

Low Diameter Random Decomposition (LDRD)

with parameter δ is a randomized algorithm which creates partition $C_1, \dots, C_\ell$ of V s.t.

1) $\text{diam}(C_i) \leq \delta \quad \forall i \in [\ell],$

2) $\Pr[u, v \text{ in different clusters}] \leq O(\log n) \cdot \frac{d(u, v)}{\delta} \quad \forall u, v \in V$

Sps we have LDRD with parameter $1-\epsilon$.

Apply to (V, d) to get $C_1, \dots, C_\ell$, let $F = \bigcup_{i=1}^{\ell} \delta(C_i)$

$\Rightarrow$ since $d(s_i, t_i) \geq 1$, s_i, t_i in different clusters

$\Rightarrow s_i, t_i$ not connected in $G \setminus F \Rightarrow F_{\text{feasible}}$

$$\text{Let } Z_e = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[c(F)] = \mathbb{E}\left[\sum_{e \in E} c(e) Z_e\right] = \sum_{e \in E} c(e) \mathbb{E}[Z_e]$$

$$\leq \sum_{e \in E} c(e) \cdot O(\log n) \frac{x_e}{1-\epsilon}$$

$$\leq O(\log n) \sum_{e \in E} c(e) x_e$$

$$= O(\log n) \cdot V^* = O(\log n) \cdot LP$$

So just need to construct LDRD.

Already know how to do this!

One level of FRT!

- Choose r uniformly at random from $[\delta/4, \delta/2]$
- Choose a permutation π of V uniformly at random
- $S = V$
- for $i = 1$ to n {
 - $P = B(\pi(i), r) \cap S$
 - IF $P \neq \emptyset$ {
 - Add P as a cluster
 - $S = S \setminus P$

Consider $u, v \in V$

$$S_w = \begin{cases} 1 & \text{if } w \text{ first vertex in } \pi \text{ s.t. } B(w, r) \cap \{u, v\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$X_w = \begin{cases} 1 & \text{if } |B(w, r) \cap \{u, v\}| = 1 \\ 0 & \text{otherwise} \end{cases}$$

u, v in different clusters iff $\sum_{w \in V} S_w X_w = 1$

Lemma: For each $w \in V$ $\exists b_w \in \mathbb{R}$ s.t.

1) $\Pr[S_w = 1 \mid X_w = 1] \leq b_w$, and

2) $\sum_{w \in V} b_w \leq O(\log n)$

Pf: see FRT

Lemma: $\Pr[X_w = 1] \leq \frac{4d(w, u)}{\delta} \quad \forall w \in V$

Pf: w.l.o.g., $d(w, u) \leq d(w, v)$

$$\Pr[X_w = 1] = \Pr[d(w, u) \leq r < d(w, v)]$$

$$\leq \frac{d(w, v) - d(w, u)}{\frac{\delta}{2} - \frac{\delta}{4}} \leq \frac{4d(w, u)}{\delta}$$

$$Pr[u, v \text{ in different clusters}] = Pr\left[\sum_{w \in V} s_w x_w = 1\right]$$

$$= E\left[\sum_{w \in V} s_w x_w\right] = \sum_{w \in V} E[s_w x_w]$$

$$= \sum_{w \in V} Pr[s_w = 1 \wedge x_w = 1]$$

$$= \sum_{w \in V} Pr[s_w = 1 \mid x_w = 1] Pr[x_w = 1]$$

$$\leq \sum_{w \in V} b_w \cdot \frac{4d(u, v)}{\delta}$$

$$= \frac{4d(u, v)}{\delta} \sum_{w \in V} b_w$$

$$\leq O(\log n) \cdot \frac{d(u, v)}{\delta}$$