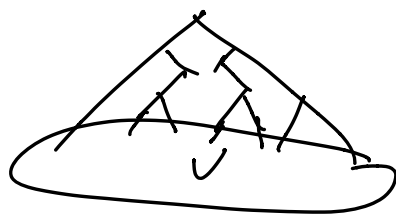


FRT Tree Embedding

Def. A **tree metric** (V', T) for a set of nodes V is a tree T with vertices V' s.t. $V \subseteq V'$ are the leaves of T , and a nonnegative length for each edge in T



So tree metric for V is tree with V as leaves,
gives distances between leaves

Then [Fakcharoenphol, Rao, Talwar '03]:

Let (V, d) be a metric space. Then there is a randomized, polytime algorithm that produces a tree metric (V', T) for V s.t.

$$1) d(u, v) \leq d_T(u, v) \quad \forall u, v \in V$$

$$2) E[d_T(u, v)] \leq O(\log n) \cdot d(u, v) \quad \forall u, v \in V$$

Embedding into a **distribution** of trees (dominating trees)

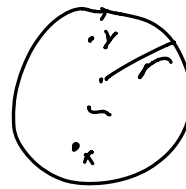
Setup:

$$- B(u, r) = \{v \in V : d(u, v) \leq r\}$$

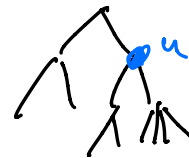
$$- \forall u \in V, \min_{v \in V: u \neq v} d(u, v) = 1 \quad (\text{rescaling})$$

$$- \text{For } S \subseteq V, \text{diam}(S) = \max_{u, v \in S} d(u, v)$$

$$- \Delta = 2^{\lceil \log \text{diam}(V) \rceil} \quad (\text{smallest power of 2 s.t. } \Delta \geq \text{diam}(V))$$



Hierarchical Cut Decomposition:



Tree metric (V', T) on V s.t.

$$1) \forall u \in V', S_u = \{v \in V : v \text{ descendant of } u \text{ in } T\}$$

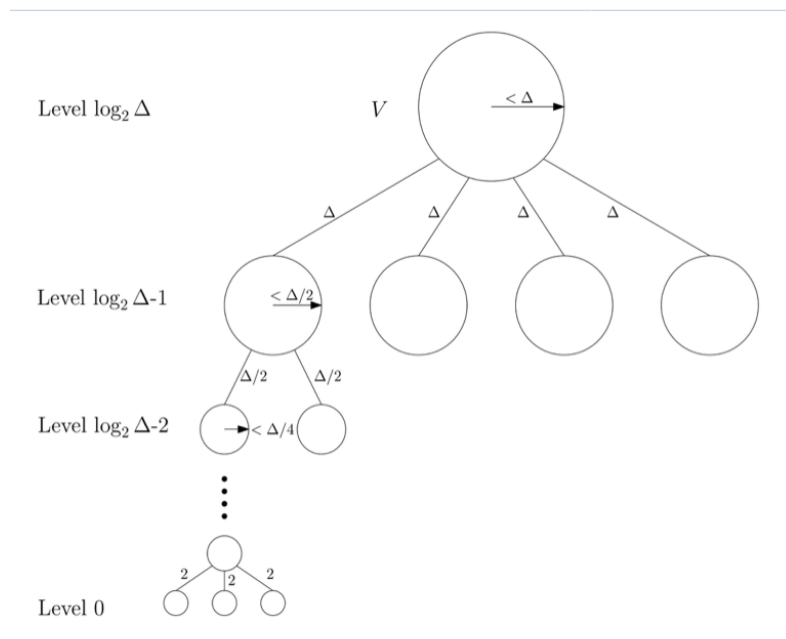
$$\Rightarrow x \in V, S_x = \{x\}$$

$$S_r = V \quad (r \text{ root of } T)$$

$$2) \text{ If } u \text{ at level } i \text{ (height } i) \text{ in } T,$$

$$\text{diam}(S_u) < 2^i$$

$$3) \text{ Length on an edge between level } i \text{ and level } i+1 \text{ is } 2^{i+1}$$



Lemma: If the LCA of $u, v \in V$ at level i , then $d_T(u, v) \leq 2^{i+2}$. Also, $d_T(u, v) \geq d(u, v) \quad \forall u, v \in V$

Pf: Let $u, v \in V$, $w = \text{LCA}(u, v)$ at level i



$$d_T(u, w) = \sum_{j=1}^i 2^j = 2^{i+1} - 2$$

$$d_T(v, w) = \sum_{j=1}^i 2^j = 2^{i+1} - 2$$

$$\Rightarrow 2^i \leq d_T(u, v) \leq 2^{i+2}$$

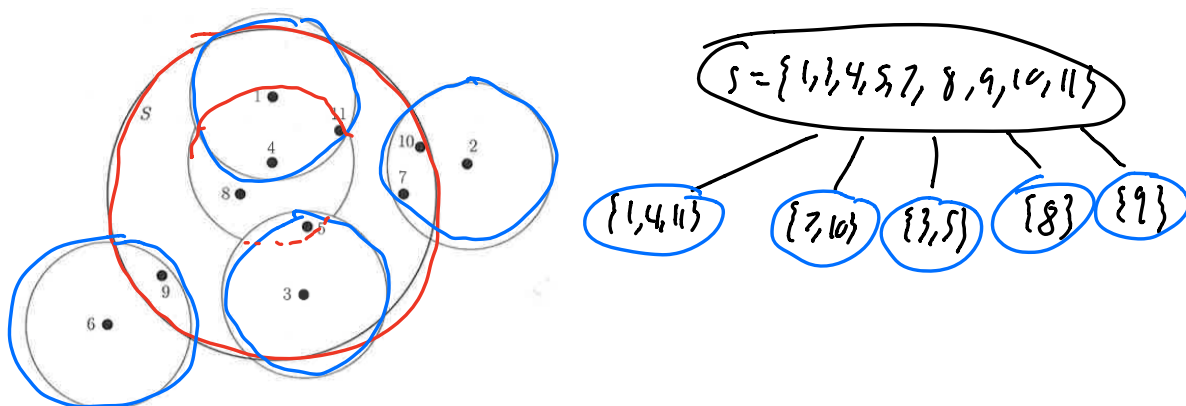
$$\Rightarrow d_T(u, v) \leq 2^{i+2} \quad \checkmark$$

$$d(u, v) \leq \text{diam}(S_w) \leq 2^i \leq d_T(u, v) \quad \checkmark$$

Note: Hierarchical cut decompositions very special tree metrics:
Hierarchically Well-Separated Tree (HST)

FRT Algorithm:

- Sample r_0 uniformly at random from $[\frac{1}{2}, 1)$
- Let $r_i = 2^i r_0$
- Sample a permutation π of V uniformly at random
- Initialize T to be only a root node r at level $\log \Delta$, $S_r = V$
- for $i = \log \Delta - 1$ down to 0 :
 - Let $\mathcal{C}$ be nodes at level $i+1$
 - for $C \in \mathcal{C}$:
 - Let $S = C$
 - for $j = 1$ to n :
 - $P = B(\pi(j), r_{i+1}) \cap S$
 - if $P \neq \emptyset$:
 - Add P as child of C at level i
 - $S = S \setminus P$



Claim: Hierarchical Cut Decomposition

pf: Let u at level i . Need to show $\text{diam}(S_u) \leq 2^i$

$$S_u = B(\pi(i), r_{i-1}) \cap S \text{ for some } S, i$$

$$\Rightarrow \text{diam}(S_u) \leq \text{diam}(B(\pi(i), r_{i-1}))$$

$$\leq 2 \cdot r_{i-1} = 2 \cdot r_0 2^{i-1} \leq 2 \cdot 2^{i-1} = 2^i \checkmark$$

$$\Rightarrow \text{Lemma implies } d(u, v) \leq d_T(u, v)$$

$$\text{Then: } \mathbb{E}[d_T(u, v)] \leq O(\log n) d(u, v) \quad \forall u, v \in V$$

Rest of class proving this.

Fix $u, v \in V$

Def: - w **settles** u, v at level i if w is the first vertex in π s.t. $B(w, r_{i-1}) \cap \{u, v\} \neq \emptyset$

- w **cuts** u, v at level i if
 $|B(w, r_{i-1}) \cap \{u, v\}| = 1$

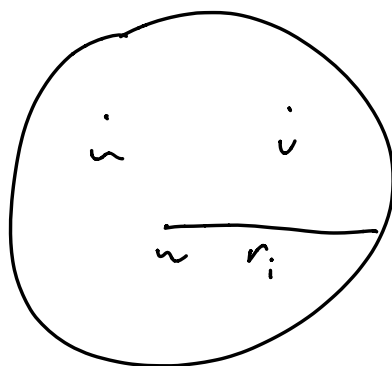
Observation: $L(A(u, v))$ is at level $i+1$ if i is the largest value s.t. the vertex w which settles u, v at level i also cuts u, v at level i

Pf sketch:



:

u, v not in same level i cluster



u, v in same level $i+1$ and above clusters

Useful Random Variables:

$$S_{iu} = \begin{cases} 1 & \text{if } u \text{ settles } u, v \text{ at level } i \\ 0 & \text{otherwise} \end{cases}$$

$$X_{iu} = \begin{cases} 1 & \text{if } u \text{ cuts } u, v \text{ at level } i \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ there is a node which both settles and cuts u, v at level i iff $\sum_{u \in V} S_{iu} X_{iu} = 1$

Let i^* be level of $L(A(u, v))$

$$\Rightarrow 1) d_T(u, v) \leq 2^{i^*+2}, \text{ and}$$

$$2) i^*-1 \text{ largest } i \text{ s.t. } \sum_{u \in V} S_{iu} X_{iu} = 1$$

$$\Rightarrow d_T(u, v) \leq 2^{i^*+2} = \max_{i: \sum_{u \in V} S_{iu} X_{iu} = 1} 2^{i+3}$$

$$\leq \sum_{i=0}^{\log \Delta} 2^{i+3} \sum_{u \in V} S_{iu} X_{iu}$$

$$= \sum_{i=0}^{\log \Delta} \sum_{u \in V} 2^{i+3} S_{iu} X_{iu}$$

$$= \sum_{w \in V} \sum_{i=0}^{\log D} 2^{i+3} S_{i,w} X_{i,w}$$

Take expectations:

$$E[d_T(u,v)] \leq E\left[\sum_{w \in V} \sum_{i=0}^{\log D} 2^{i+3} S_{i,w} X_{i,w}\right]$$

$$= \sum_{w \in V} \sum_{i=0}^{\log D} 2^{i+3} E[S_{i,w} X_{i,w}]$$

$$= \sum_{w \in V} \sum_{i=0}^{\log D} 2^{i+3} \Pr[S_{i,w} = 1 \wedge X_{i,w} = 1]$$

$$= \sum_{w \in V} \sum_{i=0}^{\log D} 2^{i+3} \Pr[S_{i,w} = 1 | X_{i,w} = 1] \cdot \Pr[X_{i,w} = 1]$$

Intuition: $X_{i,w}$ depends only on v_o , not π .

Hopefully easier to analyze

Two lemmas, show how they imply main theorem

($E[d_T(u,v)] \leq O(\log n) d(u,v)$), then prove lemmas

Lemma 1: For every $w \in V$, there is some $b_w \in \mathbb{R}_{\geq 0}$ s.t.

$$1) \Pr[S_{i,w} = 1 | X_{i,w} = 1] \leq b_w \quad \forall i, \text{ and}$$

$$2) \sum_{w \in V} b_w \leq O(\log n)$$

Lemma 2: $\sum_{i=0}^{\log \Delta} 2^{i+3} \Pr[X_{i,u} = 1] \leq 32 d(u,v) \quad \forall u \in V$

(continue previous inequalities)

$$\begin{aligned}
 E[d_T(u,v)] &\leq \sum_{u \in V} \sum_{i=0}^{\log \Delta} 2^{i+3} \Pr[S_{i,u} = 1 \mid X_{i,u} = 1] \cdot \Pr[X_{i,u} = 1] \\
 &\leq \sum_{u \in V} b_u \sum_{i=0}^{\log \Delta} 2^{i+3} \Pr[X_{i,u} = 1] \\
 &\leq \sum_{u \in V} b_u 32 d(u,v) \\
 &= 32 d(u,v) \sum_{u \in V} b_u \\
 &\leq O(\log n) \cdot d(u,v)
 \end{aligned}$$

pf of Lemma 1:

want to analyze $\Pr[S_{i,u} = 1 \mid X_{i,u} = 1]$.

Order V by distance to $\{u,v\}$:

$$d(w_j, \{u,v\}) \leq d(w_{j+1}, \{u,v\}) \quad \forall j$$

sps $X_{i,w_j} = 1 \Rightarrow |\{B(w_j, r_{i-1}) \cap \{u,v\} \mid w_j \leq w_i\}| = 1$

$$\Rightarrow |B(w_k, r_{i-1}) \cap \{u, v\}| \geq 1 \quad \forall k < j$$

$$\Rightarrow \text{if } w_k \text{ earlier in } \pi \text{ than } w_j, S_{i, w_j} = 0$$

$$\Pr[w_j \text{ earlier than } w_k \text{ in } \pi \quad \forall k < j] = \frac{1}{j} \quad (\text{random permutation})$$

$$\text{Set } b_{w_j} = \frac{1}{j}$$

$$\Rightarrow \Pr[S_{i, w_j} = 1 \mid X_{i, w_j} = 1] \leq \frac{1}{j} = b_j$$

$$\sum_{w \in V} b_w = \sum_{j=1}^n b_{w_j} = \sum_{j=1}^n \frac{1}{j} = H_n = O(\log n)$$

PF of Lemma 2:

$$\underline{wTS} : \sum_{i=0}^{\log D} 2^{i+3} \Pr[X_{i, w} = 1] \leq 32 d(w, v)$$

$$w \text{ LCA}, d(w, u) \leq d(w, v)$$

$$X_{i, w} = 1 \Leftrightarrow \text{circle diagram} \quad d(w, u) \leq r_{i-1} \leq d(w, v)$$


r_{i-1} distributed uniformly in $[2^{i-2}, 2^{i-1})$

$$\Rightarrow \Pr[X_{i,w} = 1] = \Pr[r_{i-1} \in [d(w,w), d(w,v)]]$$

$$= \frac{|[2^{i-2}, 2^{i-1}) \cap [d(w,w), d(w,v))|}{|[2^{i-2}, 2^{i-1})|}$$

$$= \frac{|[2^{i-2}, 2^{i-1}) \cap [d(w,w), d(w,v))|}{2^{i-2}}$$

$$\Rightarrow 2^{i+3} \Pr[X_{i,w} = 1] = \frac{2^{i+3}}{2^{i-2}} |[2^{i-2}, 2^{i-1}) \cap [d(w,w), d(w,v))|$$

$$= 32 |[2^{i-2}, 2^{i-1}) \cap [d(w,w), d(w,v))|$$

$$\Rightarrow \sum_{i=0}^{\log \Delta} 2^{i+3} \Pr[X_{i,w} = 1] \leq$$

$$\leq \sum_{i=0}^{\log \Delta} 32 |[2^{i-2}, 2^{i-1}) \cap [d(w,w), d(w,v))|$$

$$= 32 |[d(w,w), d(w,v))|$$

$$= 32 (d(w,v) - d(w,w)) \quad \swarrow \quad d(w,w) \leq d(w,w) + d(w,v)$$

$$\leq 32 d(w,v)$$

