Integer Multicommodity Flow:

Input: - $G=(V,E)$

- $(s_1,t_1), (s_2,t_2), \dots, (s_k,t_k)$

Feasible: paths $p_1, p_2, \dots, p_k$ s.t. $p_i \in P_i$

P_i: all $s_i \to t_i$ paths in G

Objective: $\min \max_{e \in E} \text{cong}(e)$

$\text{cong}(e) = |\{i \in [k] : e \in p_i\}|$

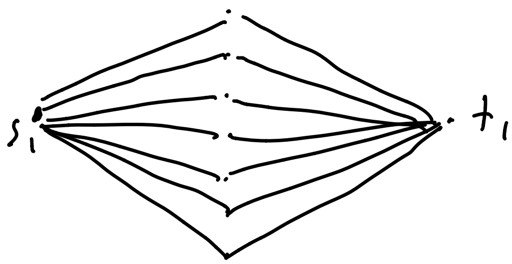
LP Relaxation:

$$
\begin{aligned}
\min \quad & W \\
\text{s.t.} \quad & \sum_{p \in P_i} x_p = 1 && \forall i \in [k] \\
& \sum_{i=1}^{k} \sum_{p \in P_i : e \in p} x_p \le W && \forall e \in E \\
& 0 \le x_p \le 1 && \forall i \in [k], \forall p \in P_i
\end{aligned}
$$

<u>Problem</u>: Too many variables!

<u>Problem</u>: Integrality gap $\geq n-2$

<u>Pf</u>:



$OPT = 1$

LP:

$x_p = \frac{1}{n-2} \quad \forall P$

$W = \frac{1}{n-2}$

Algorithm: [Raghavan-Thompson '87]

- Solve LP to get (x^*, W^*)
- For each $i \in [k]$,
  - $\{x^*_p\}_{p \in P_i}$ form a distribution
  - Choose P_i from this distribution

Thm: With high probability, $ALG \le O(\log n) \cdot OPT$

not LP!

Note: can improve to $O\left(\frac{\log n}{\log\log n}\right)$

Def: Random variables:

- $Z_P = \begin{cases} 1 & \text{if path } P \text{ chosen} \\ 0 & \text{otherwise} \end{cases}$

- $Y_e^i = \begin{cases} 1 & \text{if } P_i \text{ (path chosen for } i) \text{ uses } e \\ 0 & \text{otherwise} \end{cases}$

- $Y_e = \sum_{i=1}^{k} Y_e^i = \#$ chosen paths using e $= \text{cong}(e)$

Lemma: $E[Y_e] \le w^* \quad \forall e \in E$

Pf:

$$E[Y_e] = E\left[\sum_{i=1}^{k} Y_e^i\right]$$

$$= E\left[\sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i : e \in P} Z_P\right] \qquad \left(Y_e^i = \sum_{P \in \mathcal{P}_i : e \in P} Z_P\right)$$

$$= \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i : e \in P} E[Z_P]$$

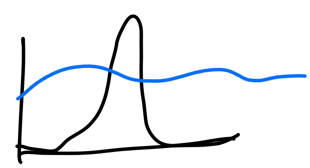
$$= \sum_{i=1}^{k} \sum_{p \in P_i : e \in p} x_p^* \qquad (E[Z_p] = x_p^*)$$

$$\le w^* \qquad (\text{LP constraint})$$

Not done! Proved $\max_{e \in E} E[Y_e] \le w^*$

Not the same as $E[\max_{e \in E} Y_e]$

Need concentration of measure: close to expectation with high probability



Thm [Chernoff Bounds]:

Let $X_1, \dots, X_n$ be independent random variables distributed over $[0, 1]$ (not necessarily uniformly).

Let $X = \sum_{i=1}^{n} X_i$. Note: $E[X] = \sum_{i=1}^{n} E[X_i]$

- $\forall\ 0 < \varepsilon < 1$: $\Pr[X > (1+\varepsilon) E[X]] \le e^{-\frac{\varepsilon^2}{3} E[X]}$

$$\Pr[X < (1-\varepsilon) E[X]] \le e^{-\frac{\varepsilon^2}{2} E[X]}$$

- $\forall t > 2e\,E[X]$, $Pr[X > t] \le 2^{-t}$

Use Chernoff to analyze algorithm:

<u>Claim</u>: $Pr[Y_e > 3\log n \cdot \max(w^*, 1)] \le \frac{1}{n^3}$

<u>Pf</u>:

$$3\log n \cdot \max(w^*, 1) > 2e \cdot w^* \ge 2e\,E[Y_e]$$

$n \ge 4 \Rightarrow 3\log n > 2e$

Apply Chernoff:

$$Pr[Y_e > 3\log n \cdot \max(w^*, 1)]$$

$$\le 2^{-3\log n \cdot \max(w^*, 1)}$$

$$\le 2^{-3\log n}$$

$$= \frac{1}{n^3}$$

✓

Union bound over edges:

$$\Pr\left[\max_{e\in E} Y_e > 3\log n \cdot \underbrace{\max(w^*, 1)}_{\le OPT}\right] \le n^2 \cdot \frac{1}{n^3} = \frac{1}{n}$$

$\Rightarrow$ With high probability, $\max_{e\in E} \text{cong}(e) \le O(\log n) \cdot OPT$

<u>Extensions</u>:

- Better version of Chernoff: $O\left(\frac{\log n}{\log\log n}\right)$

- If w^* larger, can do better!

<u>Ex</u>: Sps $w^* \ge \frac{18e}{\varepsilon^2} \ln n$

If $E[Y_e] \ge \frac{9}{\varepsilon^2} \ln n$

$$\Rightarrow \Pr[Y_e > (1+\varepsilon)E[Y_e]] \le e^{-\frac{\varepsilon^2}{3}E[Y_e]} \le e^{-3\ln n} = \frac{1}{n^3}$$

If $E[Y_e] < \frac{9}{\varepsilon^2} \ln n$

$\Rightarrow w^* \ge 2e\, E[Y_e]$

$$\Rightarrow \Pr[Y_e > W^*] \le 2^{-W^*} \le \frac{1}{n^3}$$

$\Rightarrow (1+\epsilon)$- approx w.h.p.

Solving the LP:

LP intuition: flow along paths

Main idea:

solve equivalent compact representation (polynomial size)

x_{uv}^i : flow from $s_i \to t_i$ using edge $u \to v$

$$\min \quad W$$

$$\text{s.t.} \quad \sum_{v \in N(u)} x_{uv}^i - \sum_{v \in N(u)} x_{vu}^i = 0 \qquad \forall i \in [k], \forall u \notin \{s_i, t_i\}$$

$$\sum_{v \in N(s_i)} x_{s_i v}^i - \sum_{v \in N(s_i)} x_{v s_i}^i = 1 \qquad \forall i \in [k]$$

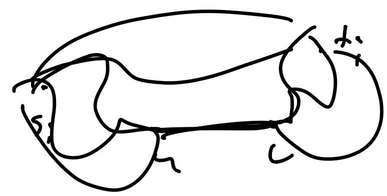
$$\sum_{i=1}^{k} (x_{uv}^i + x_{vu}^i) \le W \qquad \forall \{u,v\} \in E$$

$$0 \leq x_{uv}^i \leq 1 \qquad \forall i \in [k], \forall \{u,v\} \in E$$

Poly size!

<u>Lemma</u>: Let $(\{x_p\}, W)$ be solution to original LP. There is a solution to compact LP with same cost W.

<u>Pf Sketch</u>:



Let $x_{uv}^i = \sum_{p \in P_i : (u,v) \in p} x_p$.

$\Rightarrow$ All constraints satisfied $\Rightarrow$ feasible solution

<u>Lemma</u>: Let $(\{x_{uv}^i\}, W)$ be a solution to compact LP. Then there is a solution to original LP of same cost, which can be found in polynomial time

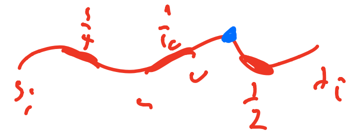
<u>Pf Sketch</u>:

"Flow-Path decomposition"

for $i = 1$ to k {

while $\exists P \in \mathcal{P}_i$ s.t. $x^i_{uv} > 0 \quad \forall (u,v) \in P$ {

- Set $x_P = \min_{(u,v) \in P} x^i_{uv}$,



- Set $x^i_{uv} = x^i_{uv} - x_P \quad \forall (u,v) \in P$

}

}

Correctness:

induction $\Rightarrow$ flow in = flow out always true,

($s_i \to t_i$ flow in x_P's) + ($s_i \to t_i$ flow in x^i_{uv}'s) = 1

Running time:

Each time find path ≥ 1 edge sets $x^i_{uv} = 0$

$\Rightarrow \leq k \cdot n^2$ iterations

Each iteration just connectivity $\Rightarrow$ poly time