Deterministic Rounding: UFL

Metric Uncapacitated Facility Location (UFL):

Input: - Metric space (V, d)

- Facility Opening costs $f: V \to \mathbb{R}^+$

Feasible solution: $S \subseteq V$ $\quad S \neq \emptyset$

Objective: $\min \ cost(S) = \sum_{i \in S} f(i) + \sum_{j \in V} \underbrace{d(j, S)}_{= \min_{x \in S} d(j,x)}$

ILP:

Variables: $y_i \quad \forall i \in V$ (intuition: 1 if $i \in S$, 0 o/w)

$x_{ij} \quad \forall i,j \in V$ (intuition: 1 if j assigned to i)

$$\min \underbrace{\sum_{i \in V} f(i) y_i}_{= F(x,y)} + \underbrace{\sum_{j \in V} \sum_{i \in V} d(i,j) x_{ij}}_{= C(x,y)} = Z(x,y)$$

$$\text{s.t.} \quad \sum_{i \in V} x_{ij} = 1 \qquad \forall j \in V$$

$$x_{ij} \le y_i \qquad \forall i,j \in V$$

$$0 \le x_{ij} \le 1 \qquad \cancel{x_{ij} \in \{0,1\}} \qquad \forall i,j \in V$$

$$0 \le y_i \le 1 \qquad \cancel{y_i \in \{0,1\}} \qquad \forall i \in V$$

Thm: This is an exact formulation

pf sketch:

Let $S \subseteq V$. Set $y_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$

$x_{ij} = \begin{cases} 1 & \text{if } j \text{ assigned to } i \text{ in } S \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow (x,y)$ feasible, and $Z(x,y) = \text{cost}(S)$

$\Rightarrow OPT(ILP) \le OPT(UFL)$

Let (x,y) feasible for ILP

set $S = \{i \in V : y_i = 1\}$

$\Rightarrow \text{cost}(S) = Z(x,y)$

$\Rightarrow OPT(UFL) \le OPT(ILP)$

LP relaxation: change

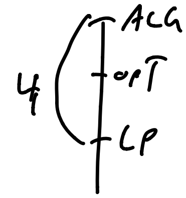
$y_i \in \{0,1\}$ to $0 \le y_i \le 1$

$x_{ij} \in \{0,1\}$ to $0 \le x_{ij} \le 1$

$\Rightarrow OPT(LP) \le OPT(ILP) = OPT(UFL)$

Can solve LP in poly time!

<u>Main theorem</u>: Given feasible fractional solution (x, y), in poly time we can find integral $(\hat{x}, \hat{y})$ s.t. $Z(\hat{x}, \hat{y}) \le 4\, Z(x, y)$

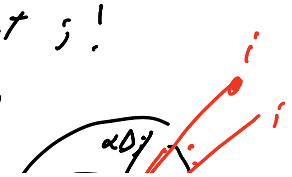


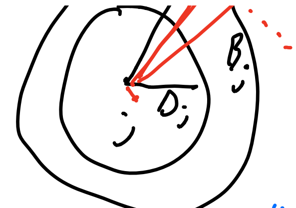
$\Rightarrow$ 4-approximation

<u>Filtering</u> (thought experiment, eventually used in analysis)

<u>Def</u>: $\Delta_j = \sum_{i \in U} d(i,j)\, x_{ij}$ (fractional connection cost of j)

- would be great for connection cost of j if we opened a facility within distance Δ_j of j!
- would that hurt facility opening costs?





<u>Def</u>: $B_j = \{ i \in V : d(i,j) \le \alpha \Delta_j \}$

parameter we set later: think 2, or $\frac{4}{3}$

<u>Claim</u>: $\sum_{i \in B_j} x_{ij} \ge \frac{\alpha - 1}{\alpha} \Leftrightarrow \sum_{i \notin B_j} x_{ij} \le \frac{1}{\alpha}$

<u>Pf</u>: Markov's inequality. More formally,

assume false $\Rightarrow \sum_{i \in B_j} x_{ij} < \frac{\alpha-1}{\alpha} \Rightarrow \sum_{i \notin B_j} x_{ij} > \frac{1}{\alpha}$

$$\Rightarrow \Delta_j = \sum_{i \in V} d(i,j) x_{ij} \ge \sum_{i \notin B_j} d(i,j) x_{ij}$$

$$\ge \sum_{i \notin B_j} \alpha \Delta_j x_{ij} = \alpha \Delta_j \sum_{i \notin B_j} x_{ij} > \Delta_j \quad \Rightarrow\Leftarrow$$

<u>Lemma</u>: Given fractional solution (x,y), can find fractional solution (x', y') s.t.

1) $F(x', y') \le \frac{\alpha}{\alpha - 1} F(x,y)$

2) If $x'_{ij} > 0$, then $i \in B_j$

Pf: Set
$$x'_{ij} = \begin{cases} 0 & \text{if } i \notin B_j \\ \dfrac{x_{ij}}{\sum_{k \in B_j} x_{kj}} & \text{if } i \in B_j \end{cases}$$

$$y'_i = \min\left(1, \frac{\alpha}{\alpha-1} y_i\right)$$

Property 2: Obvious ✓

Property 1:

$$F(x', y') = \sum_{i \in V} f(i) y'_i \leq \sum_{i \in V} f(i) \frac{\alpha}{\alpha-1} y_i$$
$$= \frac{\alpha}{\alpha-1} \sum_{i \in V} f(i) y_i = \frac{\alpha}{\alpha-1} F(x, y)$$

Feasible:

All vars in $[0, 1]$ ✓

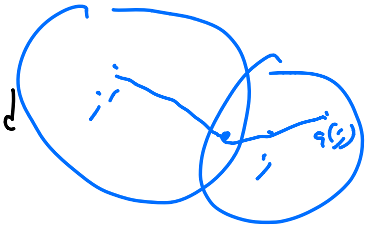
$$\sum_{i \in V} x'_{ij} = \sum_{i \in B_j} x'_{ij} = \sum_{i \in B_j} \frac{x_{ij}}{\sum_{k \in B_j} x_{kj}} = 1 \checkmark$$

$$x'_{ij} = \frac{x_{ij}}{\sum_{k \in B_j} x_{kj}} \leq \frac{x_{ij}}{\frac{\alpha-1}{\alpha}} \leq \frac{\alpha}{\alpha-1} y_i = y'_i$$

(if $\frac{\alpha}{\alpha-1} y_i < 1$)

Rounding

(Use initial LP solution (x,y), not filtered solution)



Init: $S = \emptyset$, all nodes unassigned

while $\exists$ unassigned nodes:

- Let j be unassigned node with min Δ_j
- Open $a(j) = \arg\min_{i \in B_j} f(i)$, assign j to $a(j)$
- Forall unassigned j' with $B_{j'} \cap B_j \neq \emptyset$, assign j' to $a(j)$

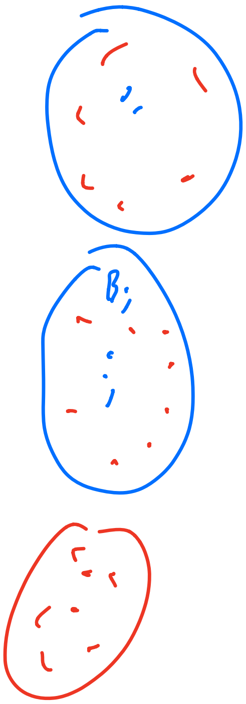
Let $\hat{S}$ be open facilities, integral solution $(\hat{x}, \hat{y})$

Lemma: $F(\hat{x},\hat{y}) \leq F(x',y') \leq \frac{\alpha}{\alpha-1} F(x,y)$

already proved

Pf:

$$F(\hat{x},\hat{y}) = \sum_{i \in V} f(i)\hat{y}_i = \sum_{j \text{ considered by ALG}} f(a(j))$$ (def of ALG)



$$\leq \sum_{\substack{j \text{ considered} \\ \text{by ALG}}} f(a(j)) \sum_{i \in B_j} y'_i$$

(filtered solution assigns j fractionally within B_j)

$$= \sum_{\substack{j \text{ considered} \\ \text{by ALG}}} \sum_{i \in B_j} f(a(j))\, y'_i$$

$$\leq \sum_{\substack{j \text{ considered} \\ \text{by ALG}}} \sum_{i \in B_j} f(i)\, y'_i$$

(def of $a(j)$)

$$\leq \sum_{i \in V} f(i)\, y'_i$$

(if j, j' considered by ALG, $B_j \cap B_{j'} = \emptyset$)

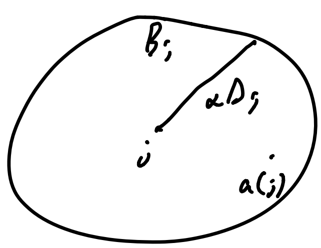
$$= F(x', y')$$

<u>Lemma</u>: $d(j, \hat{S}) \leq 3\alpha\Delta_j$

<u>Pf</u>:

<u>Case 1</u>: j considered by ALG

$$\Rightarrow a(j) \in \hat{S} \Rightarrow d(j, \hat{S}) \leq d(j, a(j)) \leq \alpha\Delta_j$$



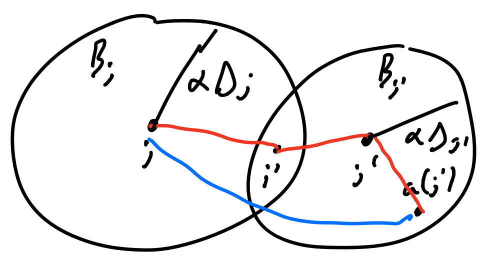
Case 2 : j not considered by ALG

$\Rightarrow$ some j' considered by ALG with

1) $\Delta_{j'} \leq \Delta_j$

2) $B_j \cap B_{j'} \neq \emptyset$

Let $i' \in B_j \cap B_{j'}$



$$\begin{aligned} d(j,\hat{S}) &\leq d(j, a(j')) \\ &\leq d(j,i') + d(i',j') + d(j', a(j')) \\ &\leq \alpha\Delta_j + \alpha\Delta_{j'} + \alpha\Delta_{j'} \\ &\leq 3\alpha\Delta_j \end{aligned}$$

$$\Rightarrow C(\hat{x},\hat{y}) = \sum_{j\in U} d(j,\hat{S}) \leq 3\alpha \sum_{j\in U} \Delta_j = 3\alpha\, C(x,y)$$

$$\Rightarrow Z(\hat{x},\hat{y}) = F(\hat{x},\hat{y}) + C(\hat{x},\hat{y})$$

$$\leq \frac{\alpha}{\alpha-1} F(x,y) + 3\alpha\, C(x,y)$$

$$\leq \max\left(\frac{\alpha}{\alpha-1}, 3\alpha\right)\left(F(x,y)+C(x,y)\right)$$

$$= \max\left(\frac{\alpha}{\alpha-1}, 3\alpha\right) Z(x,y)$$

So $\max\left(\frac{\alpha}{\alpha-1}, 3\alpha\right)$-approximation!

$$\frac{\alpha}{\alpha-1} = 3\alpha \Rightarrow \alpha = \frac{4}{3}$$

$\Rightarrow$ 4-approximation