19.1 Set Cover

Input: Universe $\mathcal{U}$, Collection $\mathcal{S} \subseteq 2^{\mathcal{U}}$, costs $c : \mathcal{S} \to \mathbb{R}^+$.

Feasible: $\mathcal{I} \subseteq \mathcal{S}$ s.t. $\bigcup_{S \in \mathcal{I}} S = \mathcal{U}$

Objective: $\min \sum_{S \in \mathcal{I}} c(S)$

19.1.1 LP for Set Cover

Primal Program:

$$\begin{aligned} \text{minimize:} \quad & \sum_{S \in \mathcal{S}} c(S) X_S && \textbf{(SC-Primal)} \\ \text{subject to:} \quad & \sum_{S : e \in S} X_S \geq 1 \quad \forall e \in \mathcal{U} && (19.1.1) \\ & X_S \geq 0 \qquad \forall S \in \mathcal{S} && (19.1.2) \end{aligned}$$

Dual Program:

$$\begin{aligned} \text{maximize:} \quad & \sum_{e \in \mathcal{U}} Y_e && \textbf{(SC-Dual)} \\ \text{subject to:} \quad & \sum_{e \in S} Y_e \leq c(S) \quad \forall S \in \mathcal{S} && (19.1.3) \\ & Y_e \geq 0 \qquad \forall e \in \mathcal{U} && (19.1.4) \end{aligned}$$

19.1.2 Greedy

Algorithm:

Keep choosing set which minimizes $\frac{c(S)}{|S \cap \mathcal{U}'|}$ where $\mathcal{U}'$ is the set of uncovered elements.

Suppose at iteration t, greedy chooses S_t which covers n_t new elements and costs greedy $c(S_t)$. Suppose greedy first covered e at time t. Set

$$Y'_e = \frac{c(S_t)}{n_t}$$

which may not be dual feasible. Note that:

$$\sum_{S \in \text{ALG}} c(S) = \sum_{e \in \mathcal{U}} Y'_e.$$

Lemma 19.1.1 $\vec{Y}$ *as given by*

$$Y_e = \frac{Y'_e}{H_n}$$

is dual feasible.

Proof: Let $S \in \mathcal{S}$. Suppose at the beginning of iteration k, a_k elements in S are uncovered. Let $A_k \subseteq S$ be the elements first covered in iteration k, hence

$$|A_k| = a_k - a_{k+1}.$$

Could have picked S, the average cost would be $\frac{c(S)}{a_k}$. Then:

$$\forall e \in A_k, \quad Y'_e \leq \frac{c(S)}{a_k}.$$

To see if $\vec{Y}$ is feasible, $\sum_{e \in S} Y_e \leq c(S)$ should hold for all $S \in \mathcal{S}$:

$$\begin{aligned}
\sum_{e \in S} Y_e &= \frac{1}{H_n} \sum_{e \in S} Y'_e \\
&= \frac{1}{H_n} \sum_{k=1}^{l} \sum_{e \in A_k} Y'_e \qquad (l \text{ is total number of iterations}) \\
&\leq \frac{1}{H_n} \sum_{k=1}^{l} \sum_{e \in A_k} \frac{c(S)}{a_k} \\
&= \frac{1}{H_n} \sum_{k=1}^{l} (a_k - a_{k+1}) \frac{c(S)}{a_k} \\
&= \frac{c(S)}{H_n} \sum_{k=1}^{l} \frac{a_k - a_{k+1}}{a_k} \\
&= \frac{c(S)}{H_n} \sum_{k=1}^{l} (\overbrace{\frac{1}{a_k} + \frac{1}{a_k} + \ldots + \frac{1}{a_k}}^{a_k - a_{k+1}}) \\
&\leq \frac{c(S)}{H_n} \sum_{k=1}^{l} (\frac{1}{a_k} + \frac{1}{a_k - 1} + \ldots + \frac{1}{a_{k+1} + 1}) \\
&\leq \frac{c(S)}{H_n} \sum_{k=1}^{|S|} \frac{1}{i} \\
&= \frac{H_{|S|}}{H_n} c(S) \leq c(S).
\end{aligned}$$

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Theorem 19.1.2 *Greedy is H_n-approximation.*

Proof:

$$\begin{aligned}
\text{greedy} &= \sum_{S \in \text{ALG}} c(S) \\
&= \sum_{e \in \mathcal{U}} Y'_e \\
&= \sum_{e \in \mathcal{U}} Y_e H_n \\
&= H_n \sum_{e \in \mathcal{U}} Y_e \\
&\leq H_n \sum_{e \in \mathcal{U}} Y^*_e && \text{(with } \vec{Y}^* \text{ the dual optimum)} \\
&\leq H_n \sum_{S \in \mathcal{S}} c(S) X^*_S && \text{(by weak duality)} \\
&\leq H_n \cdot \text{OPT}
\end{aligned}$$

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19.1.3 Primal-Dual Scheme

1. Write down primal (min) and dual (max)
2. Start with $\vec{X} = \vec{0}, \quad \vec{Y} = \vec{0}$. ($\vec{X}$ is primal infeasible, $\vec{Y}$ is dual feasible.)
3. Until $\vec{X}$ feasible:
   (a) Increase $\vec{Y}$ until some dual constraint become tight.
   (b) Select some of the tight dual constraints, increase corresponding primal variables integrally.
4. For analysis, prove $c^\top X \leq \alpha \cdot b^\top Y$ for some α.

19.1.4 PD for Set Cover

- $\vec{Y} = \vec{0}, \quad \mathcal{I} = \emptyset$.
- While $\exists e \in \mathcal{U}$ s.t. $e \notin \bigcup_{S \in \mathcal{I}} S$:
  - Increase Y_e until
  $$\exists S \notin \mathcal{I}, \quad e \in S, \quad \sum_{e' \in S} Y_{e'} = c(S)$$
  - Add S to $\mathcal{I}$
- Return $\mathcal{I}$.

It must be noted that in the first line of the While loop, we don't actually need to increase Y_e continuously, instead, we directly solve for the minimum needed increase:

$$\min_{\substack{S \notin \mathcal{I} \\ e \in S}} \left(c(S) - \sum_{e' \in S} Y_{e'} \right).$$

Theorem 19.1.3 *PD is a f-approximation* Set Cover*.*

Proof:

$$\begin{aligned}
\text{PD} &= \sum_{S \in \mathcal{I}} c(S) \\
&= \sum_{S \in \mathcal{I}} \sum_{e \in S} Y_e \\
&= \sum_{e \in S} Y_e \cdot |\{S \in \mathcal{I} : e \in S\}| \\
&\leq f \sum_{e \in S} Y_e \\
&\leq f \cdot \text{OPT} && \text{(by weak duality)}
\end{aligned}$$

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19.2 Shortest s-t Path

Input:

- Graph $G = (V, E)$
- $s, t \in V$
- $c : E \mapsto \mathbb{R}^+$

Moreover, let $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$.

Primal Program:

$$\begin{aligned}
\text{minimize:} \quad & \sum_{e \in E} c(e) X_e && \textbf{(SP-Primal)} \\
\text{subject to:} \quad & \sum_{e \in \delta(S)} X_e \geq 1 \quad \forall S \in \mathcal{S} && (19.2.5) \\
& X_e \geq 0 \qquad\quad \forall e \in E && (19.2.6)
\end{aligned}$$

Dual Program:

$$\text{maximize:} \quad \sum_{S \in \mathcal{S}} Y_S \qquad (\textbf{SP-Dual})$$

$$\text{subject to:} \quad \sum_{\substack{S \in \mathcal{S} \\ e \in \delta(S)}} Y_S \leq c(e) \quad \forall e \in E \qquad (19.2.7)$$

$$Y_S \geq 0 \qquad \forall S \in \mathcal{S} \qquad (19.2.8)$$

PD Algorithm:

- $\vec{Y} = \vec{0}, \quad F = \emptyset$.
- While no $s - t$ path in (V, F):
  - Let C be the component of (V, F) that contains s
  - Increase Y_c until

$$\exists e \in E \quad \text{s.t.} \sum_{\substack{S \in \mathcal{S} \\ e \in \delta(S)}} Y_S = c(e)$$

  - Add e to F
- Return $s - t$ path P in F.

Lemma 19.2.1 *At all steps in the PD Algorithm, F is a tree containing s.*

Proof: Proof in book. ■

Theorem 19.2.2 *PD finds a shortest path.*

Proof:

$$\begin{aligned}
\sum_{e \in P} c(e) &= \sum_{e \in P} \sum_{S: e \in \delta(S)} Y_S \\
&= \sum_{S \in \mathcal{S}} Y_S \cdot |P \cap \delta(S)| \\
&= \sum_{S \in \mathcal{S}} Y_S \leq \text{OPT}.
\end{aligned}$$

■

Claim 19.2.3 *If $Y_S > 0$, then $|P \cap \delta(S)| = 1$.*

Proof: Proof in book. ■