600.469 / 600.669 Approximation Algorithms **Lecturer:** Michael Dinitz
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18.1 LP Duality: An example

An example LP in canonical form (called the primal):

$$\begin{aligned}
\min \quad & 3x_1 + x_2 + 4x_3 \\
(I1) \text{ s.t.} \quad & x_1 + 2x_2 \geq 3 \\
(I2) \quad & x_1 + 2x_3 \geq 2 \\
(I3) \quad & 2x_1 + 3x_2 + x_3 \geq 4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

A possible solution to this LP is $x_1 = 0, x_2 = 3/2, x_3 = 1$, which gives a cost of $3(0)+(3/2)+4(1) = 11/2$. However, how do we know that 11/2 is the best answer? Our strategy is to combine constraints to get a lower bound on the objective. For instance, since x_1, x_2, x_3 are all nonnegative and any solution must satisfy constraints $(I1), (I2), (I3)$, we know that any solution must satisfy any nonnegative combination of constraints $(I2)$ and $(I3)$. For example, taking 1/2 of $(I2)$ and 1/3 of $(I3)$ implies that any solution must satisfy

$$\begin{aligned}
& 1/2(I2) + 1/3(I3) \geq 1/2(2) + (1/3)4 \\
\Leftrightarrow \quad & 1/2(x_1 + 2x_3) + 1/3(2x_1 + 3x_2 + x_3) \geq 7/3 \\
\Leftrightarrow \quad & (1/2 + 2/3)x_1 + x_2 + (1 + 1/3)x_3 \geq 7/3 \\
\Leftrightarrow \quad & 7/6x_1 + x_2 + 4/3x_3 \geq 7/3
\end{aligned}$$

Note that all of the coefficients in this expression are less than the coefficients in the objective function. Together with nonnegativity, this implies that in any solution, the objective function is at least $3x_1 + x_2 + 4x_3 \geq 7/6x_1 + x_2 + 4/3x_3 \geq 7/3$.

We can generalize this to try to find the best such lower bound on the LP. Our goal is to find coefficients y_1, y_2, y_3 for $y_1(I1)+y_2(I2)+y_3(I3)$ that give the best way of combining the inequalities in the primal. The coefficients have to ensure that $3x_1 + x_2 + 4x_3 \geq y_1(I1) + y_2(I2) + y_3(I3)$, which will hold as long as there are no more than $3, 1, 4$ copies of x_1, x_2, x_3 respectively. Namely, $y_1 + y2 + 2y_3 \leq 3$, $2y_1 + 3y_3 \leq 1$, and $2y_2 + y_3 \leq 4$. Subject to these constraints, we want to maximize the lower bound on the primal, i.e. we want to maximize $3y_1 + 2y_2 + 4y_3$. This gives the following dual LP:

$$
\begin{aligned}
\max\ & 3y_1 + 2y_2 + 4y_3 \\
\text{s.t. } & y_1 + y_2 + 2y_3 \le 3 \\
& 2y_1 + 3y_3 \le 1 \\
& 2y_2 + y_3 \le 4 \\
& y_1, y_2, y_3 \ge 0
\end{aligned}
$$

The dual is also an LP. We can solve this LP and find that the value is optimized by $y_1 = 1/2, y_2 = 2, y_3 = 0$, which gives an objective value of $3(1/2) + 2(2) + 4(0) = 11/2$. By construction, this dual objective value is a lower bound on the primal objective value.

18.1.1 Matrix notation

More generally, the primal and dual formulation can be rewritten in matrix notation, for $c, x \in \mathbb{R}^n, b \in \mathbb{R}^n, A \in R^{m\times n}$. The following sides are equivalent and give the primal. The general process is to start with the primal, then make new variables, which becomes the dual.

$$
\begin{aligned}
\min\ & c^T x \\
\text{s.t. } & Ax \ge b \\
& x \ge 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\min\ & \sum_{j=1}^{n} c_j x_j \\
\text{s.t. } & \sum_{j=1}^{n} a_{ij} x_j \ge b_i \quad \forall i \in [m] \\
& x_j \ge 0 \quad \forall j \in [n]
\end{aligned}
$$

$$
y_1, \dots, y_m \ge 0
$$

The dual is then formulated as follows.

$$
\begin{aligned}
\max\ & \sum_{j=1}^{m} b_i y_i \\
\text{s.t. } & \sum_{i=1}^{m} a_{ij} y_i \le c_j \quad \forall j \in [n] \\
& y_j \ge 0 \quad \forall i \in [m]
\end{aligned}
$$

Or equivalently

$$
\begin{aligned}
\max\ & b^T y \\
\text{s.t. } & A^T y \le c \\
& y \ge 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\min\ & (-b)^T y \\
\text{s.t. } & (-A)^T y \ge -c \\
& y \ge 0
\end{aligned}
$$

Theorem 18.1.1 *If LP D is dual of P, then P is dual of D.*

18.1.2 Weak Duality

Weak duality says that the value of the dual is a lower bound on the value of the primal LP.

Theorem 18.1.2 *For any feasible primal dual solution pair (x, y), $b^T y \leq c^T x$ (i.e., OPT(dual)$\leq$ OPT(primal)).*

Proof of Theorem 18.1.2:

$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \big(\sum_{i=1}^{m} a_{ij} y_i \big) x_j = \sum_{i=1}^{m} \big(\sum_{j=1}^{n} a_{ij} x_j \big) y_i \geq \sum_{i=1}^{m} b_i y_i$. ■

18.1.3 Strong Duality

Strong duality says that if both primal and dual LPs are feasible, then they have the same optimal value.

Theorem 18.1.3 *If x^* optimal for primal, y^* optimal for dual, then $b^T y = c^T x$ (i.e., y^* gives the best possible lower bound).*

Note: If the primal is infeasible, the dual is unbounded.

Theorem 18.1.4 *Let (x, y) be feasible (primal, dual) solution pair, then x, y are both optimal iff*

$\forall j \in [n] \sum_{i=1}^{m} a_{ij} y_i = c_j$ *iff* $x_j > 0$. *(A)*

$\forall i \in [m] \sum_{j=1}^{n} a_{ij} x_j = b_i$ *iff* $y_i > 0$. *(B)*

(A) and (B) are called the complementary slackness conditions.

Proof of Theorem 18.1.4: Assume the complementary slackness conditions hold. Then the inequalities in (Thm18.1.2) are equalities, which implies that $\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$. Since weak duality says that $\sum_{j=1}^{n} c_j x'_j \geq \sum_{i=1}^{m} b_i y'_i$ for any feasible x' and y', (x, y) must be optimal.

Assume (x, y) are optimal solutions. Then strong duality implies that $b^T y = c^T x$. This implies that the equations in (Thm 18.1.2) hold with equality, which in turn implies that the complementary slackness conditions hold. ■

18.2 Example algorithms with LP duality

18.2.1 Max-flow, min-cut

Let $P_{s,t}$ be all $s - t$ paths. Let x_p represent the flow along path $p \in P_{s,t}$.

The LP for max-flow is

$$\begin{aligned} \max \quad & \sum_{p \in P_{s,t}} x_p \\ \text{s.t.} \quad & \sum_{p \in P_{s,t} : e \in p} x_p \leq c_e \quad \forall e \in E \\ & x_p \geq 0 \quad \forall p \in P_{s,t} \end{aligned}$$

The dual of this LP is

$$\begin{aligned} \min \ & \sum_{e \in E} c_e y_e \\ \text{s.t.} \ & \sum_{e \in P} y_e \geq 1 \quad \forall P \in \mathcal{P}_{s,t} \\ & y_e \geq 0 \quad \forall e \in E \end{aligned}$$

This is clearly the min $s - t$ cut LP! As we saw in a previous class, this LP can be rounded to an integral solution without any loss in the objective function. Hence strong duality, combined with this rounding, gives a proof that the maximum $s - t$ flow equals the min $s - t$ cut.

18.2.2 Max-multicommodity flow

Input: $G = (V, E)$, capacities $c : e \to \mathbb{R}$, k source-sink pairs $(s_1, t_1), \ldots, (s_k, t_k)$.

Feasible: Some assignment of flow from each s_i to the corresponding t_i such that the total flow on edge e is less than equal to c_e for all $e \in E$.

Objective: maximize the total flow

An LP formulation of the max-multicommodity flow problem is the following. Think of x_p^i as the amount of flow along path p from s_i to t_i.

$$\begin{aligned} \max \ & \sum_{i=1}^{k} \sum_{p \in P_{s_i,t_i}} x_p^i \\ \text{s.t.} \ & \sum_{i=1}^{k} \sum_{p \in P_{s_i,t_i} : e \in p} x_p^i \leq c_e \quad \forall e \in E \\ & x_p^i \geq 0 \quad \forall i \in [k], \forall p \in P_{s_i,t_i} \end{aligned}$$

Dual:

$$\begin{aligned} \min \ & \sum_{e \in E} c_e y_e \\ \text{s.t.} \ & \sum_{e \in p} y_e \geq 1 \quad \forall i \in [k], \forall p \in P_{s_i,t_i} \\ & y_e \geq 0 \quad \forall e \in E \end{aligned}$$

Note that the dual is exactly the multicut LP! Let F^* equal the max-multicommodity flow, and let $\hat{C}$ equal the actual min-multicut and C^* be the min fractional multicut.

Theorem 18.2.1 $F^* \leq \hat{C} \leq 4(\ln(k+1))F^*$.

Proof of Theorem 18.2.1: Let (x^*, y^*) be the optimal primal dual pair and $\hat{y}$ be the optimal multicut.

We want to show that $\hat{C}$ is bounded on either side in terms of the optimal primal value F^*.

From weak duality we get that $F^* \leq C^*$. Because C^* is a relaxation of $\hat{C}$, we get that $C^* \leq \hat{C}$. Using the previously established rounding method from class for min-multicut, we know that $\hat{C} \leq 4(\ln(k+1))C^*$. Strong duality then gives us that $4(\ln(k+1))C^* = 4(\ln(k+1))F^*$.

Therefore the optimal min-multicut solution to the dual is within $4(\ln(k+1))$ of the optimal solution of max-multicommodity. ■