600.469 / 600.669 Approximation Algorithms **Lecturer:** Michael Dinitz
Topic: LPs as Metrics II: Multicut **Date:** 4/7/15
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17.1 Multicut

Definition 17.1.1 *In the Multicut problem, we are given a graph $G = (V, E)$ with costs $c : E \to \mathbb{R}^+$, and k pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of nodes. A feasible solution is a set $F \subseteq E$ such that s_i and t_i are not connected in $G \setminus F$ for all $i \in [k]$. The objective is to minimize $\sum_{e \in F} c(e)$.*

For the remainder, we will use $\mathcal{P}_i$ to denote the set of all s_i-t_i paths. The problem admits the following LP relaxation:

$$\begin{aligned} \text{minimize:} \quad & \sum_{e \in E} c(e) \cdot x_e & & \textbf{(MULTICUT-LP)} \\ \text{subject to:} \quad & \sum_{e \in p} x_e \geq 1 \quad \forall\, i \in [k], \forall\, p \in \mathcal{P}_i & & (17.1.1) \\ & 0 \leq x_e \leq 1 \quad \text{for each edge } e \in E & & (17.1.2) \end{aligned}$$

Note: As with multiway cut, we can solve this LP in polytime via ellipsoid, using shortest path (for each $\mathcal{P}_i$) to separate. For the remainder, we will use $\vec{x}$ to refer to the solution of the LP, and set $V^* = \sum_{e \in E} c(e) x_e$ as the value of the solution.

Definition 17.1.2 *Let $d : V \times V \to \mathbb{R}^+$ be the shortest path metric using the LP solution $\vec{x}$ for the edge lengths.*

Definition 17.1.3 *For subgraph G' of G, $v \in V$, $r \geq 0$, let $B_{G'}(v, r) = \{u \in V(G') : d(u, v) \leq r\}$ be the set of nodes in G' at distance at most r from v. Note that the metric d is with respect to G, not G'.*

Definition 17.1.4 *Let $\delta_{G'}(B_{G'}(s_i, r)) = \{\{u, v\} \in E(G') : u \in B_{G'}(s_i, r), v \notin B_{G'}(s_i, r)\}$ be the set of edges leaving $B_{G'}(s_i, r)$.*

To move forward, we wanted to be able to think of edges as pipes with volume: that is, given width $c(e)$ and length x_e, the volume of an edge would be $c(e)x_e$. This motivated the following definition.

Definition 17.1.5 (Volume)

$$V_{G'}(s_i, r) = \frac{V^*}{k} + \sum_{\substack{e=\{u,v\} \in E(G') \\ u,v \in B_{G'}(s_i,r)}} c(e) x_e + \sum_{\substack{e=\{u,v\} \in E(G') \\ u \in B_{G'}(s_i,r) \\ v \notin B_{G'}(s_i,r)}} c(e)(r - d(s_i, u))$$

Note: The second term above should be thought of as the volume of all edge-pipes fully inside the ball around s_i, and the third as (a lower bound for) the volume contained in $B_{G'}(s_i, r)$ of edge-pipes leaving the ball. The first term is included to make later calculations easier.

Lemma 17.1.6 (Region-Growing Lemma) *For any subgraph G', for all $i \in [k]$, we can find in polytime a value $0 \le r < \frac{1}{2}$ such that:*

$$c(\delta_{G'}(B_{G'}(s_i, r))) \le 2\ln(k+1) \cdot V_{G'}(s_i, r)$$

(Proven later in notes.) This lemma will allow us to prove useful bounds in the following sections. It's difficult to relate it directly to the LP value, so we use volume here as an intermediary.

Algorithm 1 Constructing an integer solution

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Init: G' = G, F = ∅
for i = 1 : k do
  if s_i, t_i connected in G' then
    Let r_i ∈ [0, 1/2) be the r value from the region-growing lemma.
    Let F_i = δ_{G'}(B_{G'}(s_i, r_i)).
    F ← F ∪ F_i
    Remove B_{G'}(s_i, r_i) and all incident edges from G'
  end if
end for
return F
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Theorem 17.1.7 *The output F from Algorithm 1 is feasible.*

Proof: We need to show that s_i, t_i are disconnected in $G \setminus F$ for every i. Note that $d(s_i, t_i) \ge 1$ for all $i \in [k]$ from the first LP constraint. We consider three cases:

1. If s_i and t_i are connected at iteration i, then $r_i < \frac{1}{2}$ implies $t_i \notin B_{G'}(s_i, r_i)$, so $\bigcup_{j \le i} F_j$ disconnects s_i and t_i.
2. If s_i is removed at iteration j, t_i removed at $j' \ne j$, then they are disconnected.
3. If s_i, t_i both removed at iteration $j < i$, then $d(s_i, s_j) \le r < \frac{1}{2}$, and $d(t_i, s_j) \le r < \frac{1}{2}$. It then follows from the triangle inequality that $d(s_i, t_i) \le 2r < 1$. This is a contradiction, so this case cannot occur.

This exhausts all cases. ■

Theorem 17.1.8 *$c(F) \le 4\ln(k+1)V^* \le 4\ln(k+1) \cdot OPT$.*

Proof: Let G_i be G' at the beginning if iteration i. Let $B_i = B_{G_i}(s_i, r_i)$ (the set of nodes removed on the ith iteration of the algorithm). Let $e = \{u, v\} \in \delta(B_i), u \in B_i, v \notin B_i$. Then $r_i - d(s_i, u) \le x_e$, or we would have that $d(s_i, v) \le d(s_i, u) + x_e < r_i$, contradicting $v \notin B_i$.

Note that in the algorithm, each edge is only included in at most one set F_i. It follows that:

$$c(F) = \sum_{i=1}^{k} c(F_i) \le 2\ln(k+1) \sum_{i=1}^{k} V_{G'}(s_i, r_i)$$

Applying our definition of volume, the above is then equal to:

$$2\ln(k+1)\sum_{i=1}^{k}\left(\frac{V^*}{k}+\sum_{\substack{e=\{u,v\}\in G_i\\ u,v\in B_i}}c(e)x_e+\sum_{\substack{e=\{u,v\}\in G_i\\ u\in B_i,v\notin B_i}}c(e)(r-d(s_i,u))\right)$$

We can now apply the observation $r_i - d(s_i,u) \le x_e$ from above to give that this expression is at most:

$$2\ln(k+1)\sum_{i=1}^{k}\left(\frac{V^*}{k}+\sum_{\substack{e=\{u,v\}\in G_i\\ u\in B_i \text{ or } v\in B_i}}c(e)x_e\right)$$

We now observe that because each edge appears at most once in the algorithm and is then removed, the inner sum here is at most equal to $\sum_{e\in E}c(e)x_e$, s.t. the above is at most $2\ln(k+1)(V^*+V^*)$. This concludes the proof. ■

We now need to justify our use of the Region-Growing lemma. For the remainder, let $c(r) = c(\delta_{G'}(B_{G'}(s_i,r))$ and let $V(r) = V_{G'}(s_i,r)$.

Proof of Lemma 17.1.6: We would like to show that if $r \in [0,\frac{1}{2})$ is chosen randomly, then $E\left[\frac{c(r)}{V(r)}\right] \le 2\ln(k+1)$. Order $B(s_i,\frac{1}{2})$ as $\{v_1,\ldots,v_m\}$, where $r_j = d(s_i,v_j)$, and $0 \le r_1 \le r_2 \le \cdots \le r_m = \frac{1}{2}$. We also define $r_0 = 0$ for later calculations.

For $r_j < r < r_{j+1}$, consider $\frac{d}{dr}V(r)$. Recall the formula for volume: in particular the first two terms are constant in r, so this derivative, if it existed, might be given by $\displaystyle\sum_{e\in\delta(B_{G'}(s_i,r))}c(e) = c(r)$.

In fact, $V(r)$ is *not* a continuous function, and cannot be expected to have the same directional derivative on either side of its discontinuities. However, we can gain some insight by pretending that $V(r)$ is continuous and differentiable in $[0,\frac{1}{2})$, with $\frac{d}{dr}V(r) = c(r)$. It then follows from calculus (!) that the average value of $\frac{c(r)}{V(r)}$ over $[0,\frac{1}{2})$ is:

$$\begin{aligned}
\frac{1}{1/2}\int_0^{1/2}\frac{c(r)}{V(r)}\,dr &= 2\int_0^{1/2}\frac{1}{V(r)}\cdot\frac{dV(r)}{dr}\,dr \\
\text{(differentials cancel)} &= 2\int_0^{1/2}\frac{1}{V(r)}\,dV(r) \\
&= 2(\ln(V(\tfrac{1}{2}))-\ln(V(0))) \\
&= 2\ln\left(\frac{V(1/2)}{V(0)}\right) \\
&\le 2\ln\left(\frac{V^*/k+V^*}{V^*/k}\right) = 2\ln(k+1)
\end{aligned}$$

It then would follow from the mean value theorem that there exists some $r \in [0, \frac{1}{2})$ achieving the average value. For this r we would then have $\frac{c(r)}{V(r)} \leq 2\ln(k+1)$, so that $c(r) \leq 2\ln(k+1)V(r)$ as desired.

The analysis above was based on the (false) assumption that $V(r)$ is continuous and differentiable. We will now complete the argument by discarding this assumption. In particular, note that $V(r)$ is piecewise linear and monotone increasing with discontinuities at the r_j's listed above. Then the real average value of $\frac{c(r)}{V(r)}$ over $[0, \frac{1}{2})$ is given by (with r_j^- infinitesimally smaller than r_j):

$$\begin{aligned}
\frac{1}{1/2}\sum_{j=0}^{m}\int_{r_j}^{r_{j+1}^-}\frac{c(r)}{V(r)}\,dr &= 2\sum_{j=0}^{m}\int_{r_j}^{r_{j+1}^-}\frac{1}{V(r)}\,dV(r) \\
&= 2\sum_{j=0}^{m}(\ln(V(r_{j+1}^-) - \ln(V(r_j))) \\
(V(r) \text{ increasing}) &\leq 2\sum_{j=0}^{m}(\ln(V(r_{j+1}) - \ln(V(r_j))) \\
(\text{sum telescopes}) &\leq 2(\ln(V(r_m)) - \ln(V(r_0))) \\
(r_0 = 0, r_m = \tfrac{1}{2};\text{ recall `pretend' section}) &\leq 2\ln(k+1)
\end{aligned}$$

Before, we concluded by saying that the MVT allowed us to find an r achieving the average value. Here, because $V(r)$ is increasing and $c(r)$ is constant over each $[r_j, r_{j+1})$ interval, we can say that the smallest value of $\frac{c(r)}{V(r)}$ will occur at some r_j^-. By the above, for $r = r_j^-$ we will then have that $c(r) \leq 2\ln(k+1)V(r)$, as desired. ■