600.469 / 600.669 Approximation Algorithms **Lecturer:** Michael Dinitz
Topic: Tree Embedding II **Date:** 3/31/15
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15.1 Tree Embedding II

15.1.1 The Algorithm

Algorithm 1 FRT

Input: A vertex set V and the metric d
Output: The tree T for tree embedding

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Sample r_0 uniformly in [1/2, 1)
r_i = 2^i r_0, 1 ≤ i ≤ log_2 Δ
Let π be a random permutation of V
Set Δ to be the smallest power of 2 s.t. Δ > max_{u,v} d(u, v)
L(log Δ) = {V}
for i = log Δ down to 1 do
  L(i − 1) = ∅
  for all C ∈ L(i) do
    S ← C
    for j = 1 to n do
      if B(π(j), r_{i−1}) ∩ S ≠ ∅ then
        Add B(π(j), r_{i−1}) ∩ S to L(i − 1)
        Remove B(π(j), r_{i−1}) ∩ S from S
      end if
    end for
    Create tree nodes corresponding to all sets in L(i − 1) that are subsets of C
    Join these nodes to the node corresponding to C by edge of length 2^i
  end for
end for
```

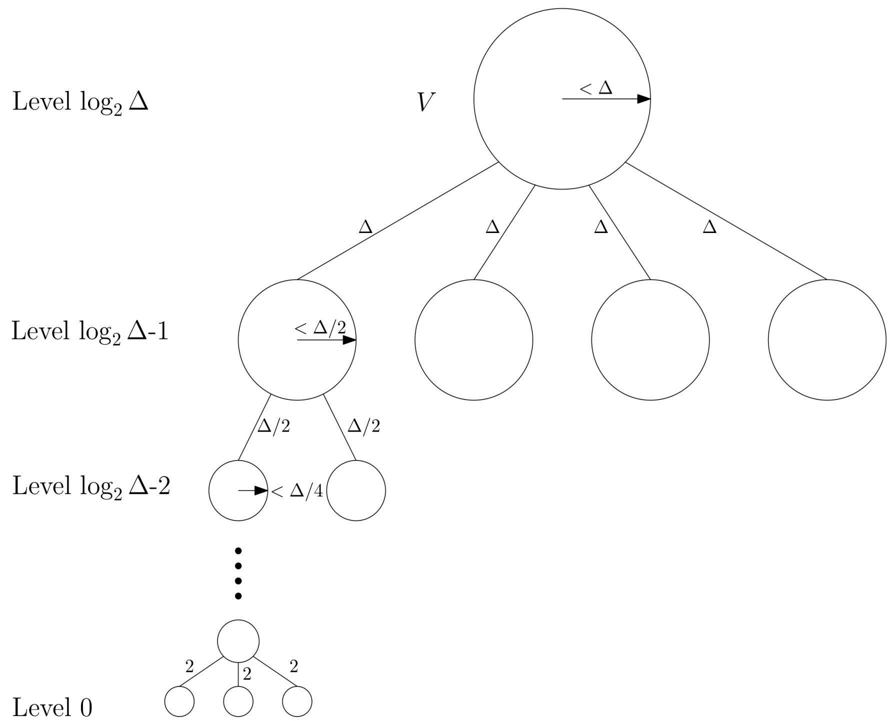

Figure 15.1.1: The tree structure (This picture is directly taken from our textbook [WS11])

15.1.2 Analysis

Theorem 15.1.1 $\forall u, v \in V, E[d_T(u,v)] \leq O(\log n) d(u,v)$.

Definition 15.1.2 *$w \in V$ settles u, v at level i, if w is the first vertex in π s.t. $B(w, r_{i-1}) \cap \{u, v\} \neq \emptyset$.*

Here $B(w, r)$ denotes the ball with center w and radius r.

Definition 15.1.3 *$w \in V$ cuts u, v at level i, if $|B(w, r_{i-1}) \cap \{u, v\}| = 1$.*

Observation 15.1.4 *$LCA(u, v)$ is at level i if i is the largest value s.t. the vertex w which settles u, v at level i also cuts u, v at level i.*

Here $LCA(u, v)$ is the least common ancestor of u, v.

Definition 15.1.5 *Define the following random variables.*

- $$S_{iw} = \begin{cases} 1, & \textit{if } w \textit{ settles } u, v \textit{ at level } i, \\ 0, & \textit{otherwise.} \end{cases} \tag{15.1.1}$$

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$$X_{iw} = \begin{cases} 1, & \text{if } w \text{ cuts } u, v \text{ at level } i, \\ 0, & \text{otherwise.} \end{cases} \tag{15.1.2}$$

Proof of Theorem 15.1.1:

$$d_T(u, v) \le \sum_{i=1}^{\log \Delta} \sum_{w \in V} 2^{i+2} S_{iw} X_{iw}$$

$$\begin{aligned} E[d_T(u, v)] &\le \sum_{i=1}^{\log \Delta} \sum_{w \in V} 2^{i+2} Pr[S_{iw} = 1, X_{iw} = 1] \\ &\le \sum_{i=1}^{\log \Delta} \sum_{w \in V} 2^{i+2} Pr[S_{iw} = 1 | X_{iw} = 1] Pr[X_{iw} = 1] \end{aligned} \tag{15.1.3}$$

Lemma 15.1.6 *Two inequalities hold:*

(1) $Pr[S_{iw} = 1 | X_{iw} = 1] \le b_w$*. Here* b *is a function on* w*.* b_w *is independent of levels.*

(2) $\sum_{w \in V} b_w \le O(\log n)$

Lemma 15.1.7 $\forall w \in V, \sum_{i=1}^{\log \Delta} 2^{i+2} Pr[X_{iw} = 1] \le 16 d(u, v)$.

By lemma 15.1.6 (1),

$$\begin{aligned} &\sum_{i=1}^{\log \Delta} \sum_{u \in V} 2^{i+2} Pr[S_{iw} = 1 | X_{iw} = 1] Pr[X_{iw} = 1] \\ &\le \sum_{i=1}^{\log \Delta} \sum_{w \in V} b_w 2^{i+2} Pr[X_{iw} = 1] \\ &= \sum_{w \in V} b_w \sum_{i=1}^{\log \Delta} 2^{i+2} Pr[X_{iw} = 1] \\ &\le \sum_{w \in V} b_w 16 d(u, v) \qquad \text{(by Lemma 15.1.7)} \\ &\le O(\log n) d(u, v) \qquad \text{(by Lemma 15.1.6 (2))} \end{aligned} \tag{15.1.4}$$

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Proof of Lemma 15.1.6:

Order the set $V = \{w_1, \ldots, w_n\}$ by distance to the pair u, v.

$$d(w_i, \{u, v\}) \le d(w_{i+1}, \{u, v\})$$

If w_j cuts u, v at level i, $|B(w_j, r_{i-1}) \cap \{u, v\}| = 1 \Longrightarrow |B(w_k, r_{i-1}) \cap \{u, v\}| > 0, \forall k \le j$.

Question 15.1.8 *If w_j settles at i, can w_k be before w_j in π for $k \leq j$?*

The answer is no, since if w_k were before w_j in π then w_k would settles u, v before w_j. Hence if $S_{iw} = 1$, then w_j is before w_k in $\pi, \forall k \leq j$. Thus we get that

$$
\begin{aligned}
&Pr[\pi(w_j) < \pi(w_k) \quad \forall k < j] = \frac{1}{j} \\
\Longrightarrow &Pr[S_{iw} = 1 | X_{iw} = 1] \leq \frac{1}{j} = b_{w_j} \\
\Longrightarrow &\sum_{j=1}^{n} b_{w_j} = \sum_{j=1}^{n} \frac{1}{j} = O(\log n)
\end{aligned}
\tag{15.1.5}
$$

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Proof of Lemma 15.1.7:

W.L.O.G, $d(w, u) \leq d(w, v)$. In order for $X_{iw} = 1$, need $r_{i-1} \in [d(w, u), d(w, v)]$.

Observation 15.1.9 *r_{i-1} is uniform in $[2^{i-2}, 2^{i-1}]$. $Pr[X_{iw} = 1] = \frac{|[2^{i-2}, 2^{i-1}) \cap [d(w,u), d(w,v)]|}{|[2^{i-2}, 2^{i-1}]|}$. Here $|[2^{i-2}, 2^{i-1}]| = 2^{i-2}$.*

So we have that

$$
\begin{aligned}
15.1.9 \Longrightarrow 2^{i+2} Pr[X_{iw} = 1] &= \frac{2^{i+2}}{2^{i-2}} |[2^{i-2}, 2^{i-1}) \cap [d(w, u), d(w, v)]| \\
&= 16 |[2^{i-2}, 2^{i-1}) \cap [d(w, u), d(w, v)]|
\end{aligned}
\tag{15.1.6}
$$

Hence

$$
\begin{aligned}
\sum_{i=1}^{\log \Delta} 2^{i+2} Pr[X_{iw} = 1] &\leq \sum_{i=1}^{\log \Delta} 16 |[2^{i-2}, 2^{i-1}) \cap [d(w, u), d(w, v)]| \\
&= 16 |[d(w, u), d(w, v)]| = 16(d(w, v) - d(w, u)) \leq 16 d(u, v)
\end{aligned}
$$

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Question 15.1.10 *If (V', T') is a tree metric for V, is there a tree metric (V, T) s.t. $d_{T'}(u, v) \leq d_T(u, v) \leq \alpha d_{T'}(u, v), \forall u, v \in V$? Here α is in $O(1)$.*

This question asks that whether we could find a tree metric without steiner nodes, i.e., so that the nodes on the tree are all in V which is the vertex set of the original graph.

Theorem 15.1.11 *[Gupta01] The answer to 15.1.10 is yes, and $\alpha = 8$.*

Here we just prove the result for the tree metric which is constructed using our tree embedding method.

Theorem 15.1.12 *If (V', T') is a tree embedding for T which is a hierarchical cut decomposition, then can find some other T s.t. $d_{T'}(u,v) \leq d_T(u,v) \leq 4d_{T'}(u,v), \forall u,v \in V$.*

Proof:

Use the following algorithm to contruct T.

(1) While $\exists$ a node $x \in V$, s.t. $p(x) \notin V$, contract $(x, p(x))$.

(2) Multiply all edge weights by 4.

Here contracting edge $(x, p(x))$ means we just merge the subtree at x into $p(x)$ and identify the newly merged node as x. Contracting makes distance go down, and hence $d_T(u,v) \leq 4d_{T'}(u,v)$. Suppose the least common ancestor of u, v is w at level i. $d_{T'}(u,v) \leq 2^{i+2}$. After contractions, their distance in T is at least 2^i (consider w and it's child). So $d_T(u,v) \geq 2^{i+2}$ as we multiply each edge weights by 4. So $d_{T'}(u,v) \leq d_T(u,v) \leq 4d_{T'}(u,v)$.

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References

Gupta01 Gupta, Anupam. "Steiner points in tree metrics don't (really) help." Proceedings of the twelfth annual ACM-SIAM symposium on Discrete algorithms. Society for Industrial and Applied Mathematics, 2001.

WS11 Williamson, David P., and David B. Shmoys. The design of approximation algorithms. Cambridge University Press, 2011.