2/22/22: (In)efficiency of Equilibria

How good are equilibria in games?

Definitions:

Cost minimization game:

- n players
- Strategy sets $S_1, S_2, \dots, S_n$

  $S = S_1 \times S_2 \times \dots \times S_n$
- Cost functions $C_i : S \to \mathbb{R}$ for each $i \in [n]$

How good/bad are the equilibria in this game?

1) What's the objective function we use to compare solutions? Need some $f: S \to \mathbb{R}$ to say how good/bad a strategy profile is

   Common choices:

   - Social cost/welfare: $f(s) = \sum_{i=1}^{n} C_i(s)$
   - min-max fairness: $f(s) = \max \{ C_i(s) : i \in [n] \}$
   - weirder: $f(s) = | \{ i \in [n] : C_i(s) < 0 \} |$

2) What kind of equilibria?

3) If multiple equilibria, which ones?

Notation: $\mathcal{M}$ = { Nash equilibria }

CCE = { coarse correlated equilibria }

$OPT = \min_{s \in S} f(s)$

Def: The Price of Anarchy is worst Nash to OPT:

$$\frac{\max_{\sigma \in \mathcal{M}} \mathbb{E}_{s \sim \sigma}[f(s)]}{OPT}$$

Def: The Price of Stability is best Nash to OPT

$$\frac{\min_{\sigma \in \mathcal{M}} \mathbb{E}_{s \sim \sigma}[f(s)]}{OPT}$$

Def: The Price of Total Anarchy is worst CCE to OPT

$$\frac{\max_{\sigma \in CCE} \mathbb{E}_{s \sim \sigma}[f(s)]}{OPT}$$

PoTS $\leq$

- PoS $\leq$ PoA $\leq$ PoTA for all games
- Can switch to utility maximization by switching min/max, inverting ratios (if want ≥ 1)

<u>Warmup</u>: Prisoner's Dilemma

	confess	silent
confess	(4,4)	(1,5)
silent	(5,1)	(2,2)

Social cost objective.

OPT: 4

Nash: 8

$\Rightarrow$ PoA = PoS = $\frac{8}{4} = 2$

Change numbers:

	confess	silent
confess	(α,α)	$(1,\alpha+1)$
silent	$(\alpha+1,1)$	(2,2)

$\Rightarrow \mathrm{OPT} = 4$

$\mathrm{Nash} = 2\alpha$

$\Rightarrow \mathrm{PoS} = \mathrm{PoA} = \frac{2\alpha}{4} = \frac{\alpha}{2}$

Really bad!

Nonatomic Routing

- Directed multigraph $G = (V, E)$ — $s, t \in V$ (source, sink)

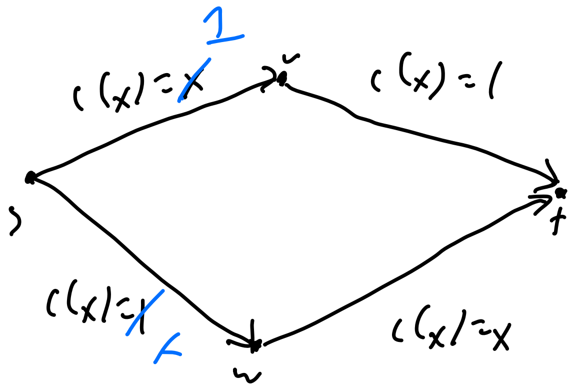
- $c_e : [0,1] \to \mathbb{R}$ for each $e \in E$

- ∞ players, or finite but super large

- strategies $= \{ s \to t \text{ paths} \} = \mathcal{P}$

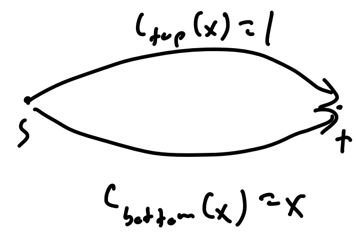
  $\Rightarrow$ strategy profile $f =$ distribution over $\mathcal{P}$ = flow

- $f_e = \sum_{p : e \in p} f_p$ (fraction of players using e), $c_p(f) = \sum_{e \in p} c_e(f_e)$ (cost of using path p)



Objective: $\sum_{P \in \mathcal{P}} c_P(f) \cdot f_P$ = average player cost

Pigou's Example:



Nash: all on bottom ; 1

OPT:

Suppose α on bottom, $1-\alpha$ on top

$\Rightarrow$ average cost $= (1-\alpha)\cdot 1 + \alpha \cdot \alpha = 1-\alpha+\alpha^2$

$\Rightarrow$ when $\alpha = \frac{1}{2} \Rightarrow \frac{3}{4}$

$$\Rightarrow PoA = PoS = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

Thm: If all edge costs are affine, then $PoA \le \frac{4}{3}$

Affine necessary!

If $c_{bottom}(x) = x^p \Rightarrow$ Nash unchanged, but

$OPT = \varepsilon$ top, $1-\varepsilon$ bottom

$= \varepsilon \cdot 1 + (1-\varepsilon)(1-\varepsilon)^p$

$= \varepsilon + (1-\varepsilon)^{p+1}$

$\to \varepsilon$ as $p \to \infty$

Network Creation Games:

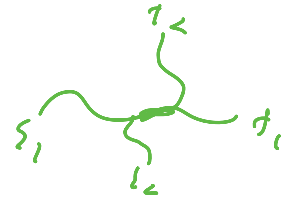
- Directed multigraph $G = (V, E)$, costs $c : E \to \mathbb{R}$
- k players, player i has source s_i, sink t_i

$$S_i = \{ s_i \to t_i \text{ paths} \} \qquad S = S_1 \times S_2 \times \dots \times S_k$$

- Given strategy profile $f = (P_1, P_2, \dots, P_k) \in S$,

$$C_i(f) = \sum_{e \in P_i} \frac{c(e)}{f_e}$$

$f_e = |\{ i \in [k] : e \in P_i \}|$

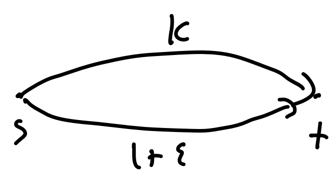


(players split costs of edges they use)

- Global objective (cost function):

$$\text{cost}(f) = \sum_{i=1}^{k} C_i(f) = \sum_{e \in \bigcup_{i=1}^{k} P_i} c(e)$$

Today: focus on pure Nash



k players, all have $s_i = s$, $t_i = t$

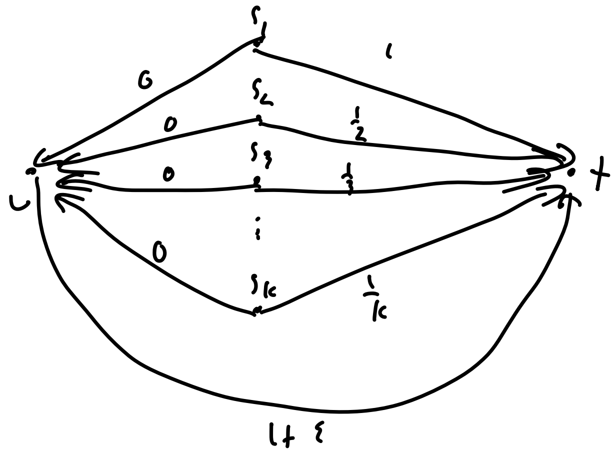
OPT: all bottom $1+\varepsilon$

Nash: all bottom $1+\varepsilon$
all top k

$\Rightarrow$ Pure price of stability $= 1$

Pure price of anarchy $= \dfrac{k}{1+\varepsilon}$

Q: Is pure PoS always 1 in network creation games?



OPT: $1+\varepsilon$

Nash: all direct

OPT: $1+\varepsilon$

Nash: $\sum_{i=1}^{k} \frac{1}{i} = H_k = \Theta(\ln k)$

$\Rightarrow$ PoS $\geq \Omega(\ln k)$

Thm: The price of stability in this type of network creation game is $O(\ln k)$

Scheduling Game:

- Players $[n]$, machines $[n]$
- Strategies = machines = $[n]$
- Cost to player is load on machine they select:

$$c_i(s) = |\{j : s_j = s_i\}|$$

- Objective: fairness = max load = makespan

$$f(s) = \max_{k \in [n]} (|\{j : s_j = k\}|)$$

OPT: 1

Pure Nash: 1

$\Rightarrow$ Pure PoA = 1

Mixed Nash:

Each player chooses machine uniformly at random

$$E[C_i(s)] = \sum_{j=1}^{n} Pr[s_j = s_i] = 1 + \sum_{j \neq i} Pr[s_j = s_i]$$

$$= 1 + (n-1) \cdot \frac{1}{n} = 1 + \frac{n-1}{n}$$

$$E[C_i(s_{-i}, k)] = 1 + \sum_{j \neq i} Pr[s_j = k] = 1 + \frac{n-1}{n}$$

$\Rightarrow$ Nash

classical result (balls-in-bins):

$$E[cost(s)] = \Theta\left(\frac{\log n}{\log\log n}\right)$$

$\Rightarrow$ Price of anarchy $= \Omega\left(\frac{\log n}{\log\log n}\right)$