2/8/22:

Before: two-player games, mixed Nash

Next couple weeks: - other kinds of games
- other kinds of equilibria

Start by "zooming in": some games where pure Nash always exist, can be found by "natural" algorithm.
- Also very natural games!

Atomic Routing Games:

- Directed graph $G=(V,E)$
- edge cost functions $c_e: \mathbb{N} \to \mathbb{R}$ for each $e \in E$
- k players
- each player $i \in [k]$ has source/sink pair $(s_i, t_i) \in V \times V$
- Player i has strategy set $S_i = \{ s_i \to t_i \text{ paths in } G \}$

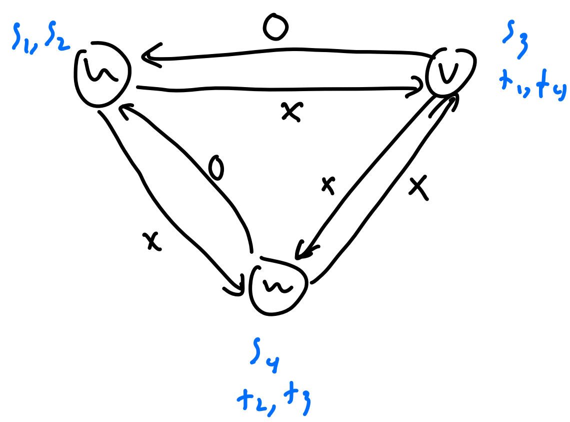
$$S = S_1 \times S_2 \times \dots \times S_k$$

- Let $f = (P_1, P_2, \dots, P_k) \in S$ be a strategy profile (flow).

Let $f_e = |\{ i \in [k] : e \in P_i \}|$ (flow through e).

$$\Rightarrow C_i(f) = \sum_{e \in P_i} c_e(f_e) \quad \text{(cost to player } i\text{)}.$$

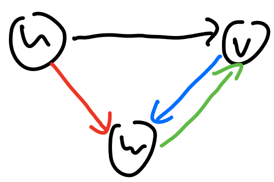
Ex:



Every player has 2 strategies:

- 1-hop path
- 2-hop path

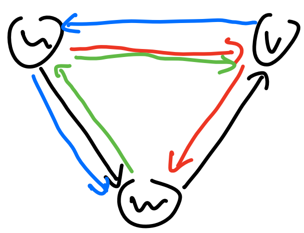
Everyone uses 1-hop path:



c_1: 1
c_2: 1
c_3: 1
c_4: 1

$\Rightarrow$ pure Nash!

Everyone uses 2-hop path:



c_1: 3
c_2: 3
c_3: 2
c_4: 2

If deviate:
c_1: 3
c_2: 3
c_3: 2
c_4: 2

Thm: Every atomic routing game has a pure Nash equilibrium

Pf: Define a potential function $\Phi : S \to \mathbb{R}$.

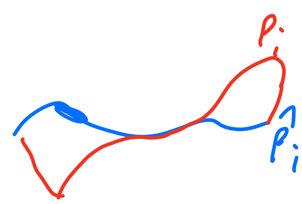
Let $f = (P_1, P_2, \dots, P_k)$

$$\Phi(f) = \sum_{e \in E} \sum_{j=1}^{f_e} c_e(j)$$

Note: Not $\sum_{i=1}^{k} C_i(f) = \sum_{i=1}^{k} \sum_{e \in P_i} c_e(f_e) = \sum_{e \in E} f_e c_e(f_e)$

Suppose player i deviates to $\hat{P}_i$

$$\Rightarrow \hat{f} = (P_1, \dots, P_{i-1}, \hat{P}_i, P_{i+1}, \dots, P_k)$$



Change to player i's cost:

$$C_i(\hat{f}) - C_i(f) = \sum_{e \in \hat{P}_i} c_e(\hat{f}_e) - \sum_{e \in P_i} c_e(f_e)$$

$$= \left(\sum_{e \in \hat{P}_i \setminus P_i} c_e(f_e + 1) + \sum_{e \in \hat{P}_i \cap P_i} c_e(f_e) \right) - \left(\sum_{e \in \hat{P}_i \cap P_i} c_e(f_e) + \sum_{e \in P_i \setminus \hat{P}_i} c_e(f_e) \right)$$

$$= \sum_{e \in \hat{P}_i \setminus P_i} c_e(f_e + 1) - \sum_{e \in P_i \setminus \hat{P}_i} c_e(f_e)$$

Change to potential:

$$\Phi(\hat{f}) - \Phi(f) = \sum_{e \in E} \sum_{j=1}^{\hat{f}_e} c_e(j) - \sum_{e \in E} \sum_{j=1}^{f_e} c_e(j)$$

$$= \sum_{e \in E} \left(\sum_{j=1}^{\hat{f}_e} c_e(j) - \sum_{j=1}^{f_e} c_e(j) \right)$$

$$= \sum_{e \in \hat{P}_i \setminus P_i} c_e(f_e + 1) - \sum_{e \in P_i \setminus \hat{P}_i} c_e(f_e)$$

$\Rightarrow$ change in potential

=

change in cost to player who deviates

Let $f = \underset{f' \in S}{\operatorname{argmin}} \ \Phi(f')$ (flow with smallest potential)

Sps player i deviates to $\hat{P}_i$ to get $\hat{f}$

$\Rightarrow C_i(\hat{f}) - C_i(f) = \Phi(\hat{f}) - \Phi(f) \geq 0$

f has min potential

$\Rightarrow$ pure Nash!

Potential Games:

Consider arbitrary one-shot simultaneous-move game:

- k players $[k]$
- Strategies S_i for $i \in [k]$ $\quad$ - $S = S_1 \times S_2 \times \cdots \times S_k$
- Costs $C_i : S \to \mathbb{R}$ for each $i \in [k]$

This is a potential game if $\exists\ \Phi : S \to \mathbb{R}$ s.t.

$$\Phi(s_{-i}, s_i') - \Phi(s) = C_i(s_{-i}, s_i') - C_i(s)$$

for all $i \in [k]$, $s \in S$, $s_i' \in S_i$

$\Rightarrow$ atomic routing games are potential games

Thm: Every potential game has a pure Nash equilibrium

pf: Already did it!

Let $s = \operatorname{argmin}_{s' \in S} \Phi(s')$

$\Rightarrow s$ Nash

"Natural Algorithm": Best response dynamics

- while s not a PNE:
  - Pick arbitrary player i and arbitrary beneficial deviation s_i'

    Let $s \leftarrow (s_{-i}, s_i')$

If BRD halts, at a PNE. But might not halt even if $\exists$ PNE!

<u>Thm</u>: In a potential game, BRD always halts (at a PNE)

<u>Pf</u>:

Every iteration: deviation beneficial to deviator

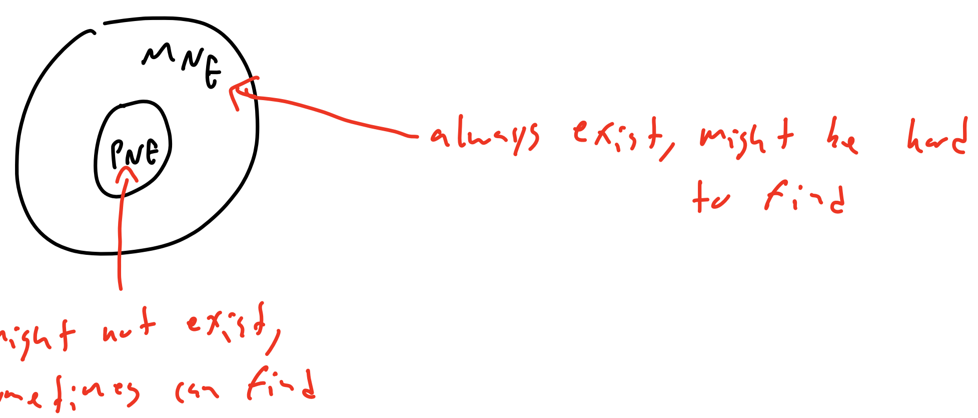
$\Rightarrow$ deviator's cost decreases

$\Rightarrow$ potential decreases

$\Rightarrow$ eventually halts since S finite

Hierarchy of Equilibria

Our world as of now:



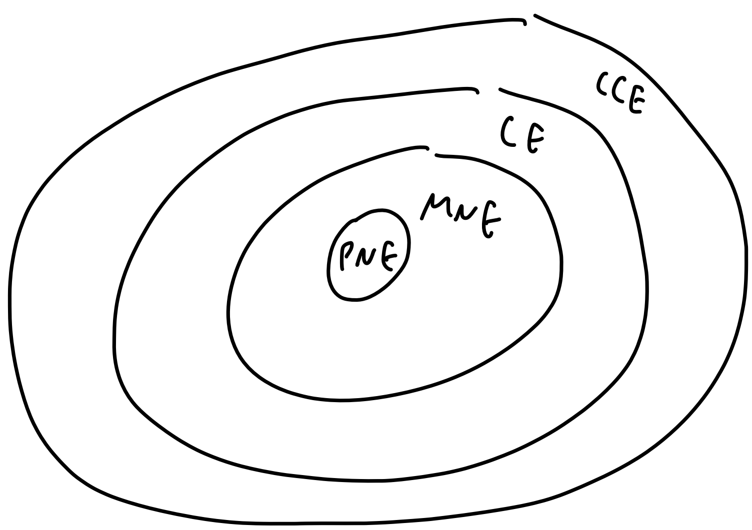
Q: Are there equilibria that always exist and can be found efficiently?

Q: What if we only allow "simple", "natural" algorithms?

YES!

Correlated equilibria: always exist, can be computed with somewhat simple algorithms

Coarse Correlated Equilibria: always exist, quite simple algorithms



Important: these definitions of equilibria predate algorithmic questions!

Ex: Street intersection game

	stop	go
stop	(0,0)	(0,−1)
go	(−1,0)	(−20,−20)

Nash: (S,G), (G,S)

- Seems unfair!
- Would like $(1/2, \frac{1}{2})$ probabilities of stop/go for each player
- Not a Nash! $Pr[go, go] = \frac{1}{4}$

Really want: $Pr[stop, go] = Pr[go, stop] = \frac{1}{2}$

- Add stoplight!
- If we're told to stop, know other player told to go
  $\Rightarrow$ don't want to deviate
- If we're told to go, know other player told to stop
  $\Rightarrow$ don't want to deviate

Cost-minimization game:

- k players $[k]$
- strategy set S_i for each player $i \in [k]$

$$S = S_1 \times S_2 \times \dots \times S_k$$

- cost functions $c_i : S \to \mathbb{R}$ for each $i \in [k]$

Write def of Nash again, but slightly differently:

<u>Def</u>: Let σ_i be a distribution over S_i $\forall i \in [k]$.

Let $\sigma = \sigma_1 \times \sigma_2 \times \dots \times \sigma_k$ be the product distribution over S induced by players' distributions.

Then σ is a mixed Nash equilibrium if

$$\mathop{E}_{s \sim \sigma} [c_i(s)] \le \mathop{E}_{s \sim \sigma} [c_i(s_{-i}, s_i')] \quad \forall i \in [k], \forall s_i' \in S_i$$

<u>Def</u>: Let σ be a distribution over $S = S_1 \times S_2 \times \dots \times S_k$.

Then σ is a correlated equilibrium if

$$\mathop{E}_{s \sim \sigma} [c_i(s) \mid s_i] \le \mathop{E}_{s \sim \sigma} [c_i(s_{-i}, s_i') \mid s_i]$$

$$\forall i \in [k], \ \forall s_i, s_i' \in S_i$$

One classical interpretation: change how game is played!

- Trusted third party $\mathcal{U}$, σ public
- $\mathcal{U}$ samples $s \sim \sigma$, keeps it secret
- $\mathcal{U}$ privately tells s_i to player i
- Each player i decides whether to play s_i or to deviate to some other strategy

$$\underbrace{\mathop{E}_{s\sim\sigma}\left[c_i(s) \mid s_i\right]}_{\text{what } i \text{ gets in expectation by playing } s_i \text{ (what it was told by } \mathcal{U}\text{)}} \le \underbrace{\mathop{E}_{s\sim\sigma}\left[c_i(s_{-i}, s_i') \mid s_i\right]}_{\text{what } i \text{ gets in expectation by deviating to } s_i'}$$

<u>Note</u>: Every Nash is a correlated equilibrium: if σ product distribution, conditioning changes nothing

Coarse Correlated Equilibria:

Def: Let σ be a distribution over $S = S_1 \times S_2 \times \dots \times S_k$.

Then σ is a coarse correlated equilibrium if

$$\mathbb{E}_{s\sim\sigma}[c_i(s)] \leq \mathbb{E}_{s\sim\sigma}[c_i(s_{-i}, s_i')] \quad \forall i \in [k], \ \forall s_i' \in S_i$$

Interpretation: have to decide whether to deviate before being told s_i

Exercise at home: If σ a CE, then σ a CCE

Intuition: if no incentive to deviate no matter what you're told to do, then no incentive to deviate even if not told what to do.

There can be CCE's that are not CEs!