2/1/22: Two-Player Games (Lemke-Howson)

Today: computing Nash in 2-player games

- Exponential time, but still interesting!
- Nash are "essentially combinatorial"
- Cool structural corollaries!

Recall: Bimatrix game

- 2 players
- $S_1 = [N]$
- $S_2 = [M]$
- $A, B \in \mathbb{R}^{N \times M}$
- $u_1(i,j) = A_{ij}$
- $u_2(i,j) = B_{ij}$

$\Rightarrow$ if $x \in \Delta_N$, $y \in \Delta_M$

$$x^T A y = \mathop{E}_{\substack{i \sim x \\ j \sim y}} [u_1(i,j)]$$

$$x^T B y = \mathop{E}_{\substack{i \sim x \\ j \sim y}} [u_2(i,j)]$$

Def: $x \in \Delta_N$ is a best response to $y \in \Delta_M$ if

$$x^T A y = \max_{x' \in \Delta_N} x'^T A y.$$

$y \in \Delta_M$ is a best response to $x \in \Delta_N$ if

$$x^T B y = \max_{y' \in \Delta_M} x^T B y'$$

$\Rightarrow$ (x, y) is a Nash if and only if x and y are best responses to each other

Lemma: Let $x \in \Delta_N$, $y \in \Delta_M$. Then x is a best response to y if and only if $\forall i \in [N]$:

$$x_i > 0 \Rightarrow (Ay)_i = \max_{k \in [N]} (Ay)_k$$

y a BR to x iff $\forall j \in [M]$:

$$y_j > 0 \Rightarrow (x^T B)_j = \max_{k \in [M]} (x^T B)_k$$

Pf sketch:

convex combinations

Simple algorithm: Support enumeration.

- If know support of Nash (x, y), can find probabilities by solving system of linear equations (Gaussian elimination).
- Try all possible supports

Skipping in lecture: details in lecture notes, NRTV 3.2

Lemke-Howson Algorithm:

Pseudo-formal: uses fun graph theory, convex geometry, linear algebra

Def: For $x \in \mathbb{R}^k$, let $S(x) = \{i : x_i \neq 0\}$ be support of x

Def: A bimatrix game is non-degenerate iff:

1) $\forall x \in \Delta_N$: $|S(x)| \geq \#$ pure best responses to x
$= |\{i \in [M] : (x^T B)_i = \max_{k \in [M]} (x^T B)_k\}|$

2) $\forall y \in \Delta_M$: $|S(y)| \geq \#$ pure best responses to y
$= |\{i \in [N] : (Ay)_i = \max_{k \in [N]} (Ay)_k\}|$

Intuition: if x a pure strategy, ≤ 1 best response
$\Rightarrow$ each row of B has unique max
if y a pure strategy, each column of A has unique max.

Notes: - Will assume all games non-degenerate
- Slightly perturbing (A, B) guarantees this
- See NRTV 3.6 for dealing with degeneracy

Use even more linear algebra: think geometrically/algebraically

Define two polyhedra:

$$\bar{P} = \{ (x, u) : x \in \Delta_N, \ \overbrace{\sum_{i=1}^{N} x_i B_{ij}}^{(x^T B)_j} \le u \quad \forall j \in [M] \}$$

$$\bar{Q} = \{ (y, v) : y \in \Delta_M, \ \underbrace{\sum_{j=1}^{M} A_{ij} y_j}_{(A y)_i} \le v \quad \forall i \in [N] \}$$

Informal: $\bar{P}$ = mixed strategies for row player + upper bound on utility for col player

$\bar{Q}$ = mixed strategies for col player + upper bound on utility for row player

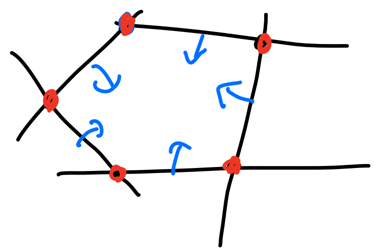
Define related polytopes: make bounded by "dividing through" by upper bound

$$P = \{ x \in \mathbb{R}^N : x_i \ge 0 \ \forall i \in [N], \ \sum_{i=1}^{N} x_i B_{ij} \le 1 \quad \forall j \in [M] \}$$

$$Q = \{ y \in \mathbb{R}^M : y_j \ge 0 \ \forall j \in [M], \ \sum_{j=1}^{M} A_{ij} y_j \le 1 \quad \forall i \in [N] \}$$

Basic facts:

- A vertex of a polyhedron/polytope is an extreme point: a point x s.t. no direction z where $x+z, x-z$ both in polyhedron/polytope
- In non-degenerate polyhedra/polytopes in $\mathbb{R}^d$, each vertex defined by exactly d tight constraints



<u>Lemma</u>: There is a bijection between vertices of $\bar{P}$ and vertices of P other than 0. Same for $\bar{Q}$ and Q.

<u>Pf sketch</u>: Let (x,u) vertex in $\bar{P}$

$\Rightarrow \left(\frac{x}{u}\right)$ vertex in P

Let $x \neq 0$ vertex in P

$\Rightarrow \left(\frac{x}{\sum_{i=1}^{N} x_i}, \frac{1}{\sum_{i=1}^{N} x_i} \right)$ vertex in $\bar{P}$

Same for Q.

Going to end up showing relationship b/w vertices and Nash

Intuition: Suppose $x \neq 0$ vertex of P

$\Rightarrow N$ tight constraints at x:

- k tight nonnegativity constraints $x_i = 0$
- $N-k$ others: values of j s.t. $\sum_{i=1}^{N} x_i B_{ij} = 1$

$\Rightarrow$ these values of j are pure best responses to x!

scaled to Δ_N

<u>Def</u>: The label set $L = [N] \uplus [M] \Rightarrow |L| = N+M$

disjoint union

Label for every pure strategy (row or column)

Let $x \in P$. $L(x) =$ labels of tight constraints

$$= \{i : x_i = 0\} \cup \{j : \sum_{i=1}^{N} x_i B_{ij} = 1\}$$

Let $y \in Q$. $L(y) =$ labels of tight constraints

$$= \{j : y_j = 0\} \cup \{i : \sum_{j=1}^{M} A_{ij} y_j = 1\}$$

$\Rightarrow |L(x)| \leq N$ (= N if vertex)

$|L(y)| \leq M$ (= M if vertex)

Thm: $(\bar{x},\bar{y}) \in \Delta_N \times \Delta_M$ is a Nash equilibrium iff

$L(x) \cup L(y) = L$, where $x \in P$ corresponds to $\bar{x}$ and
$y \in Q$ corresponds to $\bar{y}$

Pf: **If:** Sps $L(x) \cup L(y) = L$

$\Rightarrow |L(x)| = N, \quad |L(y)| = M$ (vertices)

Every label appears exactly once, either in $L(x)$ or $L(y)$

Partition $[N]$:

$N_1 = \{ i \in [N] : x_i = 0 \}$ $(i \in L(x))$

$N_2 = \{ i \in [N] : \sum_{j=1}^{M} A_{ij} y_j = 1 \}$ $(i \in L(y))$

$\Rightarrow S(x) = N_2$

$\Rightarrow \bar{x}$ BR to $\bar{y}$

Partition $[M]$:

$M_1 = \{ j \in [M] : y_j = 0 \}$ $(j \in L(y))$

$M_2 = \{ j \in [M] : \sum_{i=1}^{N} x_{ij} B_{ij} = 1 \}$ $(j \in L(x))$

$\Rightarrow S(y) = M_2$

$\Rightarrow \bar{y}$ BR to $\bar{x}$

$\Rightarrow (\bar{x}, \bar{y})$ Nash

<u>Only if:</u> Sps $(\bar{x}, \bar{y})$ Nash

Want to show: every label appears at least once in $L(x) \cup L(y)$

$(\Rightarrow L(x) \cup L(y) = L)$

Let $i \in [N]$:

- If $x_i = 0$, $i \in L(x)$
- Else $x_i > 0 \Rightarrow$ since Nash, i must be pure BR

$$\Rightarrow \sum_{j=1}^{M} A_{ij} y_j = 1$$

$$\Rightarrow i \in L(y)$$

Let $j \in [M]$:

- if $y_j = 0$, $j \in L(y)$
- else $y_j > 0 \Rightarrow$ since Nash, j must be pure BR

$$\Rightarrow \sum_{i=1}^{N} x_i B_{ij} = 1$$

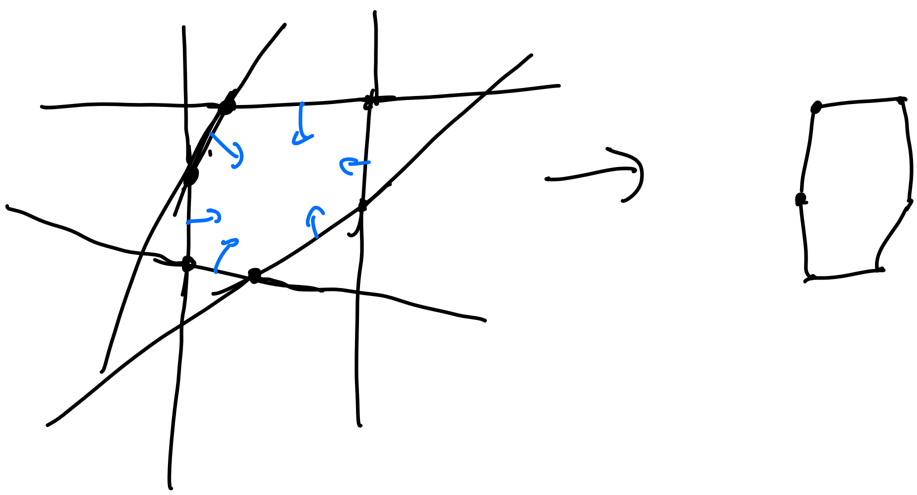
$$\Rightarrow j \in L(x)$$

✓

So to find Nash, just need to find vertices that contain all the labels!

Now switch to <u>graph theory</u>!

Given polytope, can create graph of vertices: two vertices adjacent in graph if "adjacent" (connected by 1-dim face) in polytope



For usual vertices x, x' adjacent if $|L(x) \cap L(x')| = N - \underline{1}$ (in P):

"drop" label from x, "add" label to get x'

Same for Q: $|L(x) \cap L(x')| = M - \underline{1}$

Let V_1 vertices of P

V_2 vertices of Q

Create graph G:

- $V = V_1 \times V_2$
- Edge b/w (x,y) and (x',y') if
  - x adjacent to x' in P and $y = y'$, or
  - y adjacent to y' in Q and $x = x'$

Let $k \in L$ be arbitrary label

Let $U_k = \{(x,y) \in V : L(x) \cup L(y) \supseteq L \setminus \{k\}\}$

H_k = subgraph of G induced by U_k

Thm: 1) $(0,0)$ and all Nash are in U_k, degree in H_k is 1

2) Every other vertex in U_k has degree 2 in H_k

Pf:

1) $(0,0)$ has all labels $\Rightarrow$ in U_k

(x,y) Nash $\Rightarrow L(x) \cup L(y) = L \Rightarrow$ in U_k

Degree: Let (x,y) either Nash or $(0,0)$

Sps $k \in L(x)$. (k∈L(y) will be symmetric)

$\Rightarrow$ "drop" k, move along edge in P to some x'

$\Rightarrow (x', y) \in U_k$

$\Rightarrow (x, y)$ has degree ≥ 1

If (x,y) had some other edge, would correspond to dropping some other label

$\Rightarrow$ other endpoint not in U_k

$\Rightarrow (x,y)$ has deg. 1 in H_k

2) Let (x,y) some other vertex in U_k

$\Rightarrow k \notin L(x) \cup L(y)$, or else (x,y) would have all labels $\Rightarrow (0,0)$ or Nash

$\Rightarrow \exists\, \ell \in L(x) \cap L(y)$, since $|L(x)| = N$, $|L(y)| = M$, $|L \setminus \{k\}| = N + M - 1$

$\Rightarrow$ can "drop" ℓ from x in $P \to x'$

$\Rightarrow (x', y)$ adjacent to (x,y) in H_k

can "drop" ℓ from y in $P \to y'$

$\Rightarrow (x, y')$ adjacent to (x,y) in H_k

$\Rightarrow (x,y)$ has degree ≥ 2 in H_k

Only 1 duplicated label ℓ (since in $U(k)$)

$\Rightarrow$ cannot drop any other label

$\Rightarrow$ (x,y) has degree ≤ 2 in H_k

$\Rightarrow$ (x,y) has degree $=2$ in H_k

What does this imply about structure of H_k?

H_k a collection of paths and cycles!

Lemke-Howson:

- choose $k \in L$ arbitrarily
- Start at $(0,0)$ (degree 1 in H_k), walk along path in H_k until reach a node (x,y) of degree 1
- Return (x,y)

Corollary: In any non-degenerate bimatrix game, there are an odd number of Nash equilibria