Two Player Games (Lemke-Howson)

Today: computing Nash in 2-player games
- Exponential time but still interesting!
- Nash are "essentially combinatorial"
- Cool structural corollaries!

Recall: Bimatrix game
- 2 players - \( S_1 = \mathbb{C}^N \) - \( S_2 = \mathbb{C}^N \)
- \( A, B \in \mathbb{R}^{N \times N} \)
- \( u_i(i,j) = A_{ij} \) - \( u_2(i,j) = B_{ij} \)

\( \Rightarrow \) if \( x \in \Delta_n, y \in \Delta_m \) \( x^T Ay = \max_{i \in \Delta_n} \min_{j \in \Delta_m} (i^T A j) \)
\( x^T By = \max_{i \in \Delta_n} \min_{j \in \Delta_m} (i^T B j) \)

Definition: \( x \in \Delta_n \) is a best response to \( y \in \Delta_m \) if
\( x^T Ay = \max_{i \in \Delta_n} x^T Ay \),
\( y \in \Delta_m \) is a best response to \( x \in \Delta_n \) if
\( x^T By = \max_{i \in \Delta_n} x^T By \).
$(x,y)$ is a Nash if and only if $x$ and $y$ are best responses to each other.

**Lemma:** Let $x \in \Delta_n$, $y \in \Delta_n$. Then $x$ is a best response to $y$ if and only if $orall i \in [n]$: 

$$x_i > 0 \iff (Ay)_i = \max_{k \in [n]} (Ay)_k$$

$y$ a BR to $x$ iff $\forall j \in [n]$: 

$$y_j > 0 \iff (B^T x)_j = \max_{k \in [n]} (B^T x)_k$$

**Proof sketch:**

convex combinations

**Simple algorithm:** Support enumeration.

- If know support of Nash $(x,y)$, can find probabilities by solving system of linear equations (Gaussian elimination).
- Try all possible supports.

Skipping in lecture; details in lecture notes, NRTV 3.2
**Lemma-Howson Algorithm:**

Pseudo-fundamental in graph theory, convex geometry, linear algebra.

**Def.:** For $x \in \mathbb{R}^k$, let $S(x) = \{ i : x_i \neq 0 \}$ be support of $x$.

**Def.:** A bi-matrix game is non-degenerate iff:

1. $\forall x \in D_n : |S(x)| \geq \# \text{pure best responses to } x$

   $$= \left\{|i \in E(M) : (x^T B)_i = \max_{k \in E(M)} (x^T B)_k\right\}$$

2. $\forall y \in D_m : |S(y)| \geq \# \text{pure best responses to } y$

   $$= \left\{|i \in E(N) : (A y)_i = \max_{k \in E(N)} (A y)_k\right\}$$

**Intuition:** if $x$ a pure strategy, $\leq 1$ best response

- Each row of $B$ has unique max

- If $y$ a pure strategy, each column of $A$ has unique max.

**Notes:**
- Assume all games non-degenerate.
- Slightly perturbing $(A, B)$ guarantees this.
- See NRTV 3.6 for dealing with degeneracy.

Use even more linear algebra: think geometrically/algebraically.
Define two polyhedra:

\[ \bar{p} = \{ (x, u) : x \in \Delta, \ \sum_{i=1}^n x_i B_{ij} \leq u \ \forall i \in \mathcal{M} \} \]

\[ \bar{Q} = \{ (y, u) : y \in \Delta, \ \sum_{j=1}^m A_{ij} y_j \leq u \ \forall i \in \mathcal{N} \} \]

**Internal:** \( \bar{p} \) is mixed strategies for row player + upper bound on utility for col player

\( \bar{Q} \) is mixed strategies for col player + upper bound on utility for row player

**Define related polytopes:** make bounded by “dividing through” by upper bound

\[ p = \{ x \in \mathbb{R}^n : x_i \geq 0 \ \forall i \in \mathcal{N}, \ \sum_{i=1}^n x_i B_{ij} \leq 1 \ \forall i \in \mathcal{M} \} \]

\[ q = \{ y \in \mathbb{R}^m : y_j \geq 0 \ \forall j \in \mathcal{M}, \ \sum_{j=1}^m A_{ij} y_j \leq 1 \ \forall i \in \mathcal{N} \} \]
Basic Facts:

- A vertex of a polyhedron/polytope is an extreme point: a point $x$ s.t. no direction $z$ where $x+z, x-z$ both in polyhedron/polytope.

- In non-degenerate polyhedra/polytopes in $\mathbb{R}^d$, each vertex defined by exactly $d$ tight constraints.

Lemma: There is a bijection between vertices of $\bar{P}$ and vertices of $P$ other than $0$. Same for $\bar{Q}$ and $Q$.

Proof sketch: Let $(x, w)$ vertex in $\bar{P}$

$\Rightarrow (\frac{x}{w})$ vertex in $P$

Let $x \neq 0$ vertex in $P$

$\Rightarrow (\frac{x}{\bar{x}}, \frac{1}{\bar{w}})$ vertex in $\bar{P}$

Same for $Q$. 

Intuition: Suppose \( x \neq 0 \) vertex of \( P \)

\[ \Rightarrow N \] tight constants at \( x \):
- \( k \) tight nonnegativity constants \( x_i = 0 \)
- \( N-k \) others: values of \( j \) s.t. \( \sum_{i=1}^{k} x_i \beta_{ij} = 1 \)

\( \Rightarrow \) these values of \( j \) are pure best responses to \( x \).

\[ \text{Def: The label set } L = C \cup \{ M \} \quad \text{Is disjoint union} \]

Label for every pure strategy (row or column)

Let \( x \in P \). \( L(x) = \) labels of tight constraints
\[ = \{ i : x_i = 0 \} \cup \{ i : \sum_{i=1}^{k} x_i \beta_{ij} = 1 \} \]

Let \( y \in Q \). \( L(y) = \) labels of tight constraints
\[ = \{ j : y_j = 0 \} \cup \{ i : \sum_{j=1}^{m} A_{ij} y_j = 1 \} \]

\( \Rightarrow \) \( |L(x)| \leq N \quad (= N \text{ if vertex}) \)

\( |L(y)| \leq M \quad (= M \text{ if vertex}) \)
Thm: \((\overline{x}, \overline{y}) \in \Delta_n \times \Delta_m\) is a Nash equilibrium iff

\[ L(x) \cup L(y) = \Lambda, \text{ where } x \in P \text{ corresponds to } \overline{x} \text{ and } y \in Q \text{ corresponds to } \overline{y} \]

Pr: \(\overline{x} \in P \) s.t. \( L(x) \cup L(y) = \Lambda \)

\[ \Rightarrow |L(x)| = N, \quad |L(y)| = M \quad (\text{w/d.}(\overline{x})) \]

Every label appears exactly once, either in \(L(x)\) or \(L(y)\)

\text{Partition } [CN]:

\[ N_1 = \{ i \in CN : x_i = 0 \} \quad (i \in L(x)) \]
\[ N_2 = \{ i \in CN : \sum_{j=1}^{k} A_{ij} y_j = 1 \} \quad (i \in L(y)) \]

\[ \exists \{x\} = N_2 \]

\[ \Rightarrow \overline{x} \text{ BR to } \overline{y} \]

\text{Partition } [CM]:

\[ M_1 = \{ i \in CM : y_i = 0 \} \quad (i \in L(y)) \]
\[ M_2 = \{ i \in CM : \sum_{j=1}^{k} B_{ij} x_j = 1 \} \quad (i \in L(x)) \]

\[ \exists \{y\} = M_2 \]

\[ \Rightarrow \overline{y} \text{ BR to } \overline{x} \]
\[ \exists (x, y) \text{ Nash} \]

\[ \text{Only if:} \ \exists (x, y) \text{ Nash} \]

want to show: every label appears at least once in \( L(x) \cup L(y) \)

\[ (\Rightarrow L(x) \cup L(y) = \mathcal{L}) \]

Let \( i \in \mathcal{L}(x) \):

- \( \text{If } x_i = 0, \ i \in L(x) \)
- \( \text{Else } x_i > 0 \Rightarrow \text{since Nash, } x_i \text{ must be pure BR} \)
  \[ \Rightarrow \sum_{j=1}^{m} A_{i,j} y_j = 1 \]
  \[ \Rightarrow i \in L(y) \]

Let \( j \in \mathcal{L}(y) \):

- \( \text{If } y_j = 0, \ j \in L(y) \)
- \( \text{Else } y_j > 0 \Rightarrow \text{since Nash, } j \text{ must be pure BR} \)
  \[ \Rightarrow \sum_{i=1}^{n} x_i \tau_{i,j} = 1 \]
  \[ \Rightarrow j \in L(x) \quad \checkmark \]

So to find Nash, just need to find vertices that contain all the labels!
Now switch to graph theory!

Given polytope, can create graph of vertices: two vertices adjacent in graph if "adjacent" (connected by 1-dim face) in polytope

For \( m \geq 2 \) vertices \( x, y \) adjacent if \( |L(x) \cap L(y)| = N - 1 \):

"drop" label from \( x \), "add" label to get \( x' \)

Same for \( \Omega \) : \( |L(x) \cap L(x')| = M - 1 \)

Let \( V_1 \) vertices of \( P \)

\( V_2 \) vertices of \( \Omega \)
Create graph $G$:
- $V = V_1 \times V_2$
- Edge between $(x,y)$ and $(x',y')$ if
  - $x$ adjacent to $x'$ in $P$ and $y = y'$, or
  - $y$ adjacent to $y'$ in $Q$ and $x = x'$

Let $l \in L$ be arbitrary label

Let $U_k = \{(x,y) \in V : L(x) \cup L(y) \geq L \setminus \{k\}\}$

$H_k$ is subgraph of $G$ induced by $U_k$

Then:
1) $(0,0)$ and all Nash are in $U_k$, degree in $H_k$ is 2
2) Every other vertex in $U_k$ has degree 2 in $H_k$

Proof:
1) $(0,0)$ has all labels $\rightarrow$ in $U_k$

$(x,y)$ Nash $\rightarrow$ $L(x) \cup L(y) = L \rightarrow$ in $U_k$

Degree: Let $(x,y)$ either Nash or $(0,0)$
Sps (k ∈ L(x)). (k ∈ L(y) will be symmetric)

⇒ “drop” k, move along edge in P to see x′

⇒ (x′, y) ∈ Uk

⇒ (x, y) has degree ≥ 1

If (x, y) had some other edge, would correspond to dropping some other label

⇒ other endpoint not in Uk

⇒ (x, y) has deg. 1 in Hk

2) Let (x, y) some other vertex in Uk

⇒ k ≠ L(x) ∪ L(y), or else (x, y) would have all labels ⇒ (0, 0) or Nash

⇒ ∃ l ∈ L(x) ∩ L(y), since |L(x)| = N, |L(y)| = N, |L \ {k}| = N + M - 1

⇒ can “drop” l from x in P ⇒ x′

⇒ (x′, y) adjacent to (x, y) in Hk

(can “drop” l from y in P ⇒ y′

⇒ (x, y′) adjacent to (x, y) in Hk

⇒ (x, y) has degree ≥ 2 in Hk
Only 1 duplicated label \( l \) (since in \( U_k \))
\( \Rightarrow \) cannot drop any other label
\( \Rightarrow (x, y) \) has degree \( \leq 2 \) in \( H_k \)
\( \Rightarrow (x, y) \) has degree \( = 2 \) in \( H_k \)

what does this imply about structure of \( H_k \)?

\( H_k \) a collection of paths and cycles!

Lenke-Howson:
- choose \( k \in L \) arbitrarily
- Start at \((0,0)\) (degree 1 in \( H_k \)), walk along path in \( H_k \) until reach a node \((x, y)\) of degree 1
- Return \((x, y)\)

**Corollary:** In any non-degenerate bimatrix game, there are an odd number of Nash equilibria