4/28/22: Voting Schemes and Social Choice II

Setup:

- Set A of candidates
  - Let $L = \{$total orderings of $A\}$
- Set $[n]$ of voters
  - Each $i \in [n]$ has private ordering $\succ_i \in L$

Arrow's Theorem

Def: An aggregation function is a function $F: L^n \to L$

Def: An aggregation function F is a dictatorship if $\exists\, i \in [n]$ s.t. $F(\succ_1, \succ_2, \dots, \succ_n) = \succ_i \quad \forall (\succ_1, \dots, \succ_n) \in L^n$

Def: F satisfies unanimity if following holds:

For all $(\succ_1, \dots, \succ_n)$ where $a \succ_i b \quad \forall i \in [n]$,

$$a \succ_{F(\succ_1, \dots, \succ_n)} b$$

Def: F satisfies independence of irrelevant alternatives (IIA) if following holds $\forall a,b \in A$. Let $(\succ_1, \dots, \succ_n)$ and $(\succ_1', \dots, \succ_n') \in L^n$ s.t. $\succ_i$ and $\succ_i'$ have same ordering of a,b $\forall i \in [n]$. Then a,b have same ordering in $F(\succ_1, \dots, \succ_n)$ and $F(\succ_1', \dots, \succ_n')$

Thm [Arrow]: If $|A| \geq 3$, then every aggregation function that satisfies unanimity and IIA is a dictatorship.

Gibbard-Satterthwaite:

Def: A social choice function is a function
$$f : L^n \to A$$

Def: A social choice function f is a dictatorship if $\exists i \in [n]$ s.t. $\forall (\succ_1, \dots, \succ_n) \in L^n$,
$$f(\succ_1, \dots, \succ_n) = a \Rightarrow a \succ_i b \quad \forall b \in S \setminus \{a\}$$

Def: A social choice function f is incentive-compatible if $\forall i \in [n]$, $\forall (\succ_1, \succ_2, \dots, \succ_n) \in L^n$, $\forall \succ_i' \in L$,

$$f(\succ_1, \succ_2, \dots, \succ_n) \succ_i f(\succ_1, \dots, \succ_{i-1}, \succ_i', \succ_{i+1}, \dots, \succ_n)$$

Def: f is monotone if $\forall i \in [n]$, $\forall (\succ_1, \dots, \succ_n) \in L^n$, $\forall \succ_i' \in L$:

$$f(\succ_1, \dots, \succ_n) = a \neq a' = f(\succ_1, \dots, \succ_{i-1}, \succ_i', \succ_{i+1}, \dots, \succ_n)$$

$$\Rightarrow a \succ_i a' \text{ and } a' \succ_i' a$$

(If choice switches because one player changes, that player must have switched to preferring a' to a)

Lemma: f is incentive-compatible if and only if f monotone

Pf: monotone $\Rightarrow$ IC: fix player i. By monotonicity, bidding truth is best

IC $\Rightarrow$ monotone: Switching from $\succ_i$ to $\succ_i'$ can't help $\Rightarrow$ monotone

Thm: Let f be an incentive compatible social choice function which is surjective with $|A| \geq 3$. Then f is a dictatorship.

Prove using Arrow's theorem.

Let f surjective, incentive compatible social choice fn.

__Def:__ Let $\succ \in L$ and $S \subseteq A$. Let $\succ^S \in L$ as follows:

- if $a, b \in S$ or $a, b \notin S$, then $a \succ^S b$ iff $a \succ b$
- if $a \in S$ and $b \notin S$, then $a \succ^S b$

(Move S to top of ordering)

__Claim:__ $\forall (\succ_1, \succ_2, \dots, \succ_n) \in L^n$ and $S \subseteq A$,

$$f(\succ_1^S, \succ_2^S, \dots, \succ_n^S) \in S$$

__Pf:__ Let $a \in S$

Since f surjective, $\exists (\succ_1', \succ_2', \dots, \succ_n')$ s.t.

$f(\succ_1', \dots, \succ_n') = a$

For $i \in \{0, 1, \dots, n\}$, let $P_i = (\succ_1^S, \succ_2^S, \dots, \succ_i^S, \succ_{i+1}', \dots, \succ_n')$

__Claim:__ $f(P_i) \in S \quad \forall i$

Base case: $f(P_0) = a \in S$

Induction: Sps $f(P_i) \in S$.

By monotonicity, $f(P_{i+1}) \succ_{i+1}^S f(P_i) \leftarrow \in S$

$\Rightarrow f(P_{i+1}) \in S$

$\Rightarrow f(P_n) = f(\succ_1^S, \succ_2^S, \dots, \succ_n^S) \in S$

Let $F(\succ_1, \succ_2, \dots, \succ_n)$ be defined by

$$a \succ_{F(\succ_1,\dots,\succ_n)} b \quad \text{iff} \quad f(\succ_1^{\{a,b\}}, \succ_2^{\{a,b\}}, \dots, \succ_n^{\{a,b\}}) = a$$

<u>Lemma</u>: F is an aggregation function

<u>Pf</u>: Need to show $F(\succ_1, \dots, \succ_n) \in L$ (total ordering)

<u>1</u>) Consider $a, b \in S$. Since $f(\succ_1^{\{a,b\}}, \dots, \succ_n^{\{a,b\}}) \in \{a,b\}$

by previous claim, either $a \succ_{F(\succ_1,\dots,\succ_n)} b$ or

$b \succ_{F(\succ_1,\dots,\succ_n)} a$

2) Transitivity: Sps cycle $a_1 \succ a_2 \succ \dots \succ a_k \succ a_1$

$\succ = \succ_{F(\succ_1,\dots,\succ_n)}$

Let $S = \{a_1, \dots, a_k\}$

WLOG $f(\succ_1^S, \succ_2^S, \dots, \succ_n^S) = a_2 \in S$

$\forall i \in [k]: \; P_i = (\succ_1^{\{a_1,a_2\}}, \dots, \succ_i^{\{a_1,a_2\}}, \succ_{i+1}^S, \dots, \succ_n^S)$

$f(P_0) = a_2$

induction: $f(P_i) = a_2$ (monotonicity)

$\Rightarrow f(P_n) = a_2$

$\Rightarrow a_2 \succ a_1$ (Contradiction)

<u>Lemma</u>: If f is not a dictatorship then F satisfies unanimity and IIA and is not a dictatorship

<u>Pf</u>: F not dictatorship:

f not dictatorship

Unanimity:

Sps $a \succ_i b \quad \forall i \in [n]$

$\Rightarrow \forall i, \ (\succ_i^{\{a,b\}})^{\{a\}} = \succ_i^{\{a,b\}}$

$\Rightarrow f(\succ_1^{\{a,b\}}, \dots, \succ_n^{\{a,b\}}) = f((\succ_1^{\{a,b\}})^{\{a\}}, \dots, (\succ_n^{\{a,b\}})^{\{a\}})$

$= a$ (by claim)

$\Rightarrow a \succ_{F(\succ_1, \dots, \succ_n)} b$ ✓

IIA:

Let $(\succ_1, \dots, \succ_n)$ and $(\succ_1', \dots, \succ_n') \in L^n$ s.t. $\succ_i$ and $\succ_i'$ have same ordering of a, b $\forall i \in [n]$.

$\Rightarrow$ ordering in $F(\succ_1, \dots, \succ_n)$ determined by

$f(\succ_1^{\{a,b\}}, \succ_2^{\{a,b\}}, \dots, \succ_n^{\{a,b\}})$

Ordering in $F(\succ_1', \dots, \succ_n')$ determined by

$$f\left(\succ_1^{\prime\{a,b\}}, \dots, \succ_n^{\prime\{a,b\}}\right)$$

Same induction and monotonicity argument:

S_n $\quad f\left(\succ_1^{\{a,b\}}, \succ_2^{\{a,b\}}, \dots, \succ_n^{\{a,b\}}\right) = a$

$$P_i = \left(\succ_1^{\prime\{a,b\}}, \dots, \succ_i^{\prime\{a,b\}}, \succ_{i+1}^{\{a,b\}}, \dots, \succ_n^{\{a,b\}}\right)$$

$\Rightarrow f(P_0) = a$

$f(P_i) = a \Rightarrow f(P_{i+1}) = a$ by monotonicity

$\Rightarrow f(P_n) = a$ ✓