4/26/22: Voting Schemes and Social Choice

Idea: If "vote" = "bid", "outcome" = "winner", then general mechanism design ≈ elections!

Setup:

- Set A of candidates
  - Let $L = \{$total orderings of $A\}$
- Set $[n]$ of voters
  - Each $i \in [n]$ has private ordering $\succ_i \in L$

Q1: Who should win election?

Q2: Is there a reasonable "group ordering" of A?

warmup: $|A| = 2$.

$\Rightarrow$ Majority wins

what if $|A| \geq 3$?

Def: "x beats y in a pairwise election" if majority prefers x to y: $|\{i \in [n] : x \succ_i y\}| \geq \frac{n}{2}$

Def: x is a Condorcet Winner if x beats y in a pairwise election $\forall y \in A \setminus \{x\}$

$A = \{a, b, c\}$ $n = 3$

1	2	3
a	c	b
b	a	c
c	b	a

What if a Condorcet winner does exist?

$A = \{a, b, c, d\}$ $n = 3$

1	2	3
b	c	d
a	a	a
c	d	b
d	b	c

Not even fixed by Single Transferable Vote!

Arrow's Theorem

Def: An aggregation function is a function $f: L^n \to L$

Def: An aggregation function F is a dictatorship if
$\exists\, i \in [n]$ s.t. $F(\succ_1, \succ_2, \dots, \succ_n) = \succ_i \quad \forall (\succ_1, \dots, \succ_n) \in L^n$

Properties we might want in an aggregation function:

Not a dictatorship.

Def: F satisfies unanimity if following holds:
For all $(\succ_1, \dots, \succ_n)$ where $a \succ_i b \quad \forall i \in [n]$,

$$a \succ_{F(\succ_1, \dots, \succ_n)} b$$

"Relative rankings of a and b depends only on their relative rankings in the n voter orderings."

<u>Def</u>: F satisfies independence of irrelevant alternatives (IIA) if following holds $\forall a,b \in A$. Let $(\succ_1, \dots, \succ_n)$ and $(\succ_1', \dots, \succ_n') \in L^n$ s.t. $\succ_i$ and $\succ_i'$ have same ordering of a,b $\forall i \in [n]$. Then a,b have same ordering in $F(\succ_1, \dots, \succ_n)$ and $F(\succ_1', \dots, \succ_n')$

<u>Thm [Arrow]</u>: If $|A| \geq 3$, then <u>every</u> aggregation function that satisfies unanimity and IIA is a dictatorship.

Today: assume $A = \{a, b, c\}$

Let $F: L^n \to L$ satisfy unanimity and IIA.

<u>WTS</u>: F dictatorship

<u>start</u>: find "pivotal voter for b over a" k_{ba}:

$p^0, \dots, p^n$: in p^i, voters $1, \dots, i$ prefer b to a
$i+1, \dots, n$ prefer a to b

By IIA, whether $a \succ_{F(p^i)} b$ or $b \succ_{F(p^i)} a$ does not depend on anything about c

Unanimity: $a \succ_{F(p^0)} b$ and $b \succ_{F(p^n)} a$

$\Rightarrow \exists\, k \in [n]$ s.t. $a \succ_{F(p^{k-1})} b$, $b \succ_{F(p^k)} a$

"pivotal voter k_{ba}"

<u>Lemma</u>: k_{ba} is a "partial dictator" for <u>b over c</u>:

If $p' = (\succ_1, \dots, \succ_n)$ and $b \succ_{k_{ba}} c$, then $b \succ_{F(p')} c$

<u>Pf</u>:

Let p be profile:

$1, \dots, k_{ba}-1$	$k_{ba}, \dots, n$
b	a
c	b
a	c

$\Rightarrow p = p^{k_{ba}-1}$

$\Rightarrow F(p)$: $a \succ_{F(p)} b \succ_{F(p)} c$

def of k_{ba} unanimity

Let $S \subseteq [n] \setminus \{k_{ba}\}$

Profile p^S: S = voters who switch c, b

$[k_{ba}-1] \setminus S$	$[k_{ba}-1] \cap S$	k_{ba}	$\{k_{ba}+1, \dots, n\} \setminus S$	$\{k_{ba}+1, \dots, n\} \cap S$
b	c	b	a	a
c	b	a	b	c
a	a	c	c	b

$\Rightarrow p^S = p^{k_{ba}}$

$\Rightarrow b \succ_{F(p^S)} a$ (def of k_{ba})

All voters have same ordering of a, c in p^S and p

$\Rightarrow a \succ_{F(p^S)} c$ (IIA)

$\Rightarrow p^S: \quad b \succ_{F(p^S)} a \succ_{F(p^S)} c$

$\Rightarrow$ No matter who prefers c to b, if k_{ba} prefers b to c the aggregate prefers b to c

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Nothing special about k_{ba} being dictator for b over c.

k_{bc} dictator for b over a

k_{ab} dictator for a over c

k_{cb} dictator for c over a

k_{ac} dictator for a over b

k_{ca} dictator for c over b

Inequalities:

$$k_{bc} \leq k_{ba}$$

pivotal voter for b over c must be before dictator for b over c

$$k_{ba} \leq k_{cb}$$

dictator for b over c must be before pivotal voter for c over b

similarly, $k_{cb} \leq k_{ca} \leq k_{bc}$

$\Rightarrow$ $k_{bc} = k_{ba} = k_{cb} = k_{ca}$

similar for k_{ab}, k_{ac}

$\Rightarrow$ all partial dictators the same

$\Rightarrow$ one dictator!

Gibbard - Satterthwaite:

Def: A social choice function is a function

$$f : L^n \to A$$

Thm: Let f be an incentive compatible social choice function which is surjective with $|A| \geq 3$. Then f is a dictatorship.

Prove using Arrow's theorem.

Def: Let $\succ \in L$ and $S \subseteq A$. Let $\succ^S \in L$ as follows:

- if $a,b \in S$ or $a,b \notin S$, then $a \succ^S b$ iff $a \succ b$
- if $a \in S$ and $b \notin S$, then $a \succ^S b$

Let f surjective, incentive compatible social choice fn.

Let $F(\succ_1, \succ_2, \ldots, \succ_n)$ be defined by

$$a \succ_{F(\succ_1, \ldots, \succ_n)} b \quad \text{iff} \quad f(\succ_1^{\{a,b\}}, \succ_2^{\{a,b\}}, \ldots, \succ_n^{\{a,b\}}) = a$$

<u>Lemma</u>: F is an aggregation function

<u>Lemma</u>: If f is not a dictatorship then F satisfies unanimity and IIA and is not a dictatorship