Online Auctions

Ex: Selling two identical items with time windows, unit demands

<table>
<thead>
<tr>
<th>bidder</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td>100</td>
<td>80</td>
<td>60</td>
</tr>
<tr>
<td>arrival</td>
<td>12pm</td>
<td>12pm</td>
<td>1pm</td>
</tr>
<tr>
<td>departure</td>
<td>2pm</td>
<td>2pm</td>
<td>2pm</td>
</tr>
</tbody>
</table>

with no time windows:

Allocation: bidder 1 and 2
- charged 60

How to sell online?

Natural idea: sell 1 item at 12:59 pm, 1 item at 1:59 pm

⇒ at 12:59, bidder 1 gets item for 80
  1:59, bidder 2 gets item for 60

Incentive Compatible?
- So bidder 1 lies, bids 65
  → loses first auction but wins second, pays 60!
- So bidder 1 lies about arrival, shows up at 1 pm
  → wins second auction, pays 60!

**Setup:**
- Each agent i (e.g., has private:
  - arrival time $a_i \in \mathbb{R}_{20}$
  - departure time $d_i \in \mathbb{R}_{20}$
  - valuation $v_i \in \mathbb{R}_{20}$
- Selling 1 item
- Bid: triple $(a_i, d_i, b_i)$
  - Revealed to auctioneer at time $a_i$
  - $a_i > 0$
- Mechanism: online allocation rule + price (collected when item sold)
- Utility of $i$: $v_i - \text{price paid (if gets item in } [a_i, d_i])$
- Incentive compatibility: bidding truth is dominant
Goals:
- Welfare maximization: Close to \( \max V_i \)?
- Revenue maximization:
  - All valuations drawn from same unknown \( F \) (prior-free)
  - Close to second-largest valuation?

Warmup:
- All \( [c_i, d_i] \) intervals disjoint
- No incentives

Almost like prophet inequality!
- In \( P_2 \), distributions different but known
  - Here distributions same but unknown

Instead, use secretary problem:
- Interview \( n \) job applicants in random order
- When interview applicant, find their value
- Irrevocable decision whether to hire
- Good enough for our setting: draw \( n \) valuations from \( F \), randomly permute = draw one at a time from \( F \).
There is an algorithm which gets max value with probability \( \frac{1}{2} \)

Today: \( p = \frac{1}{4} \)

**Algorithm:**
- Let \( \frac{n}{2} \) applicants go by, \( p = \text{max value} \)
- Hire next applicant with value \( \geq p \)

If best in last \( \frac{n}{2} \), second best in first \( \frac{n}{2} \):

\[ p \left[ \text{best in second half, second best in first half} \right] \geq \frac{1}{4} \]

Back to real setting (incentives, overlapping intervals)

**Attempt 2:**
- Wait until receive \( \frac{n}{2} \) bids (time \( t \))
- Set \( p = \text{largest} x \)
- Sell to next bidder above \( p \)

Not IC!
**Fix.**
- Let $t$ be time receive 2nd bid
- $p = q$ two largest bids so far
- If I active bidder w/bid $b$ sell at price $q$
- Sell to next agent above $p$ at price $p$

**Incentive Compatibility:**

**Departures:**
Leaving early can't help, leaving late makes difference only if set item after $t$: $\Rightarrow$ no value

**Values:**
- Critical bid: conditioned on getting item, bid does not affect price
  $\Rightarrow$ If $v_i$ would win overbidding does not affect price, underbidding either makes no difference or doesn't win
- If $v_i$ would lose, underbidding makes no difference, overbidding either makes no difference or $\Rightarrow$ negative utility
Arrival: Fit all other bids.

\[ r = \frac{n-1}{2} \text{ arrival time} \]
\[ s = \frac{n}{2} \text{ arrival time} \]

Item win with \( a_i \) (or else \( a_i \) at least as good):
\[ a_i > s \]
- \( a_i \) pushes \( i \) even later \( \Rightarrow \) since \( a_i \) wins at \( a_i \), \( a_i \) wins at \( a_i \)

same price

\[ r < q < s; \quad a_i < s \]
- No difference btw \( a_i, a_i' \)
- \( a_i' > s \)
- Item is better, but price \( p \) at \( a_i \) instead of \( q \) at \( a_i \)

\[ a_i < r \]
- No difference

\[ r < a_i < s \]
- \( t = a_i \) instead of \( t = r \), but \( i \) wins at price \( q \) in both
- \( a_i > s \)
- \( i \) wins at price \( p \), but \( a_i \) wins at price \( q \)
Quality:

Thm (Social Welfare): Agent with max value wins the
item with probability $\geq \frac{1}{2}$

PE: Case 1: item sold at time $t$

$\Rightarrow$ item sold to highest bid in first $\frac{1}{2}$
$\Rightarrow$ highest overall with probability $\frac{1}{2} > \frac{1}{4}$

Case 2: item sold after time $t$

$\Rightarrow$ Secretary analysis!

Thm (Revenue): $E[\text{Revenue}] = \frac{1}{2} \cdot E[\text{Revenue of Vickrey}]$

$\Rightarrow$ $E[\text{Revenue of Vickrey}] = E[2 \text{nd highest valuation}]$

Case 1: item sold at time $t$

$\Rightarrow$ sold to highest bidder in first $\frac{1}{2}$

$\Rightarrow$ with prob. $\frac{1}{4}$, highest and second highest bidders

overall win in first $\frac{1}{2}$

$\Rightarrow$ get same revenue as Vickrey
Case 2: Item sold after time $t$ just like secretary!

Generalization to $k$ items:

Technically difficult, conceptually easy

1) "Learning": do nothing until time $t$ arrives
   at $\frac{k}{2}$ agent

2) "Transition": Sell up to $\lceil \frac{k}{2} \rceil$ items at time
   $t$ to active agents with bids above $(\lceil \frac{k}{2} \rceil + 1)^{st}$
   highest so far

3) "Accepting": Set $p$ to $\lceil \frac{k}{2} \rceil$ highest bid in first half.
   Sell to any bidder above $p$ at price $p$ while supply lasts.