4/14/22: Combinatorial Auctions II

Setting: general combinatorial auction

- Players $[n]$ - m items M
- Outcome: allocation of items to bidders, where bidder i gets $S_i \subseteq M$ and $S_i \cap S_j = \emptyset$ $\forall i \neq j$

  $\Rightarrow$ # outcomes $= (n+1)^m$
- Valuations: each i has valuation fn $v_i : 2^M \to \mathbb{R}_{\geq 0}$:

  $v_i(S)$ = value of receiving bundle S

  Assume: $v_i(\emptyset) = 0$,

  Today: ignore size-of-bid issues. Assume valuation functions/bids can be efficiently represented ("bidding languages")
- Utility = value − price
- Social welfare of $(S_1, S_2, \ldots, S_n)$ is $\sum_{i=1}^{n} v_i(S_i)$

Major issue: computational hardness

Inspiration from approximation algorithms/operations research:

write an integer linear program (IP), relax to give a linear programming relaxation (LP)

ILP Formulation:

Vars $x_{i,S}$ $\forall i \in [n]$, $S \subseteq M$.

Interpretation: $x_{i,S} = \begin{cases} 1 & \text{if } S \text{ is bundle assigned to } i \\ 0 & \text{otherwise} \end{cases}$

$$\max \quad \sum_{i=1}^{n} \sum_{S \subseteq M} v_i(S)\, x_{i,S}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \sum_{S : j \in S} x_{i,S} \le 1 \qquad \forall j \in M$$

$$\sum_{S \subseteq M} x_{i,S} \le 1 \qquad \forall i \in [n]$$

$$x_{i,S} \in \{0,1\} \qquad \forall i \in [n], \forall S \subseteq M$$

LP relaxation: replace $x_{i,S} \in \{0,1\}$

by $x_{i,S} \ge 0$

Exponential #vars, LP can still be solved efficiently (NRTV 11.5.2)

- Ellipsoid with separation oracle on dual

LP OPT $\ge$ IP OPT $=$ OPT social welfare

So if LP is "integral" (has integral optimum), then LP OPT $=$ OPT social welfare, can find efficiently!

Dual LP:

Vars $p_j \ \forall j \in M$, $u_i \ \forall i \in [n]$

$$\min \sum_{i=1}^{n} u_i + \sum_{j \in M} p_j$$

$$\text{s.t.} \quad u_i + \sum_{j \in S} p_j \geq v_i(S) \qquad \forall i \in [n], \forall S \subseteq M$$

$$u_i \geq 0 \qquad \forall i \in [n]$$

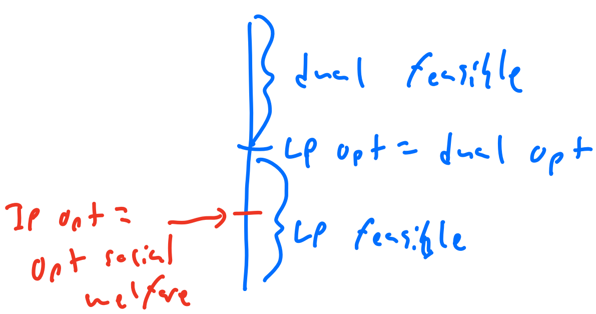
$$p_j \geq 0 \qquad \forall j \in M$$

Strong duality: if x LP feasible, (u,p) dual feasible, then

1) $$\sum_{i=1}^{n} \sum_{S \subseteq M} v_i(S)\, x_{i,S} \leq \sum_{i=1}^{n} u_i + \sum_{j \in M} p_j$$

2) If x LP optimal, (u,p) dual optimal, then

$$\sum_{i=1}^{n} \sum_{S \subseteq M} v_i(S)\, x_{i,S} = \sum_{i=1}^{n} u_i + \sum_{j \in M} p_j$$



Complementary slackness:

Let x LP feasible, (u,p) dual feasible.

Then x LP optimal, (u,p) dual optimal iff

1) $\forall i \in [n]$: either $u_i = 0$ or $\sum_{S \subseteq M} x_{i,S} = 1$ (or both)

2) $\forall j \in M$: either $p_j = 0$ or $\sum_{i=1}^{n} \sum_{S : j \in S} x_{i,S} = 1$ (or both)

3) $\forall i \in [n], \forall S \subseteq M$: either $x_{i,S} = 0$ or

$u_i + \sum_{j \in S} p_j = v_i(S)$ (or both)

Walrasian Equilibria: "competitive equilibria" in markets.

-Linear, market-clearing prices

(price for each item so demand = supply)

Def: Given item prices p_j $\forall j \in M$, bundle T is a demand for bidder i if

$$v_i(S) - \sum_{j \in S} p_j \leq v_i(T) - \sum_{j \in T} p_j \qquad \forall S \subseteq M$$

Def: Nonnegative prices p_j^* $\forall j \in M$ and allocation $S_1^*, S_2^*, \dots, S_n^*$ of the items is a Walrasian Equilibrium if

1) For every $i \in [n]$, S_i^* is a demand for i at prices p^*,

2) $p_j^* = 0$ if j not allocated $\left(j \notin \bigcup_{i=1}^{n} S_i^*\right)$

Connection: A walrasian equilibrium exists iff LP is integral

LP integral $\Rightarrow$ equilibrium maximizes social welfare!

Thm: Let p_j^* $\forall j \in M$, S_i^* $\forall i \in [n]$ be a Walrasian equilibrium. Then $\forall$ feasible LP solutions x^*,

$$\sum_{i=1}^{n} v_i(S_i^*) \geq \sum_{i=1}^{n} \sum_{S \subseteq M} v_i(S)\, x_{i,S}^*$$

$\Rightarrow$ S^* integral optimal $\Rightarrow$ maximizes social welfare, LP integral

Pf: $\forall i \in [n]$, $\forall S \subseteq M$, $v_i(S_i^*) - \sum_{j \in S_i^*} p_j^* \geq v_i(S) - \sum_{j \in S} p_j$

(def of Walrasian equilibrium)

$$\Rightarrow v_i(S_i^*) - \sum_{j \in S_i^*} p_j^* \geq \sum_{S \subseteq M} x_{i,S}^* \left(v_i(S) - \sum_{j \in S} p_j\right)$$

($\sum_{S \subseteq M} x_{i,S}^* \leq 1$ by LP constraint)

$$\Rightarrow \sum_{i=1}^{n} \left(v_i(S_i^*) - \sum_{j \in S_i^*} p_j^* \right) \geq \sum_{i=1}^{n} \sum_{S \subseteq M} x_{i,S}^* \left(v_i(S) - \sum_{j \in S} p_j^*\right)$$

$$\Rightarrow \sum_{i=1}^{n} v_i(S_i^*) \geq \sum_{i=1}^{n} \sum_{S \subseteq M} v_i(S)\, x_{i,S}^* + \sum_{i=1}^{n} \sum_{j \in S_i^*} p_j^* - \sum_{i=1}^{n} \sum_{S \subseteq M} x_{i,S}^* \sum_{j \in S} p_j^*$$

$$= \sum_{i=1}^{n} \sum_{S \subseteq M} v_i(S)\, x_{i,S}^* + \sum_{j \in M} p_j^* - \sum_{i=1}^{n} \sum_{S \subseteq M} x_{i,S}^* \sum_{j \in S} p_j^*$$

(def of Walrasian equilibrium)

$$= \sum_{i=1}^{n} \sum_{S \subseteq M} v_i(S)\, x_{i,S}^* + \sum_{j \in M} p_j^* - \sum_{j \in M} p_j^* \left(\sum_{i=1}^{n} \sum_{S : j \in S} x_{i,S}^* \right)$$

$$\geq \sum_{i=1}^{n} \sum_{S \subseteq M} v_i(S)\, x^*_{i,S} + \sum_{j \in M} p^*_j - \sum_{j \in M} p^*_j \qquad \text{(LP constraint)}$$

$$= \sum_{i=1}^{n} \sum_{S \subseteq M} v_i(S)\, x^*_{i,S}$$

<u>Thm</u>: If an integral optimal solution to LP exists, then a Walrasian equilibrium exists

<u>Pf</u>: Let x^* integral optimal LP solution

$\Rightarrow$ defines an allocation $S^*_1, S^*_2, \dots, S^*_n$

Let p^*, u^* optimal dual solution

<u>WTS</u>: $p^*_j \; \forall j \in M$, $S^*_1, S^*_2, \dots, S^*_n$ a Walrasian equilibrium

<u>S^*_i is a demand for i at prices p^*</u>:

$x^*_{i,S^*_i} = 1 \Rightarrow x^*_{i,S^*_i} > 0$

$\Rightarrow u^*_i + \sum_{j \in S^*_i} p^*_j = v_i(S^*_i)$ (complementary slackness)

$\Rightarrow v_i(S^*_i) - \sum_{j \in S^*_i} p^*_j = u^*_i$

Consider some $S \subseteq M$

$\Rightarrow \quad u_i^* + \sum_{j \in S} p_j^* \geq v_i(S)$ (dual constraint)

$\Rightarrow \quad v_i(S) - \sum_{j \in S} p_j^* \leq u_i^*$ ✓

$p_j^* = 0$ if j not allocated $\left(j \notin \bigcup_{i=1}^{n} S_i^*\right)$:

j not allocated $\Rightarrow \sum_{i=1}^{n} \sum_{S: j \in S} x_{i,S}^* = 0$

$\Rightarrow \quad \sum_{i=1}^{n} \sum_{S: j \in S} x_{i,S}^* < 1$

$\Rightarrow \quad p_j^* = 0$ (complementary slackness)