Combinatorial Auction:

- Players \( \{n\} \) - \( m \) items \( M \)
- Outcome: allocation of items to bidders, where bidder \( i \) gets \( s_i \) \( \subseteq M \) and \( s_i \cap s_j = \emptyset \) \( \forall i \neq j \)

\[ \Rightarrow \text{outcome} = (n+1)^m \]
- Valuations: each \( i \) has valuation \( \forall n \), \( v_i : 2^M \rightarrow \mathbb{R}_{\geq 0} \):

\[ v_i(s) = \text{value of receiving bundle } s \]

Assume: \( v_i(\emptyset) = 0 \)

\[ v_i(s) \leq v_i(T) \quad \forall s \subseteq T \]
- Utility = value - price
- Social welfare of \( (s_1, s_2, ..., s_n) \) is \( \sum_{i=1}^{n} v_i(s_i) \)
Simple case: "Single-Minded Bidders"

Each player $i$ has private set $T_i \leq M$ and private $v_i \in \mathbb{R}^+$. 

$$v_i(s) = \begin{cases} v_i & \text{if } T_i \subseteq s \\ 0 & \text{otherwise} \end{cases}$$

Bid: $(b_i, s_i)$

Greedy Mechanism:

Allocation Rule:

Sort and reindex so \( \frac{b_1}{\sqrt{|S_1|}} \geq \frac{b_2}{\sqrt{|S_2|}} \geq \ldots \geq \frac{b_n}{\sqrt{|S_n|}} \)

$W = \emptyset$

for $i = 1$ to $n$

\[ \text{if } (S_i \cap \bigcup_{j \neq i} S_j) = \emptyset \]

\[ \text{add } i \text{ to } W \]

\[ \text{give } S_i \text{ to } i \]

else give $\emptyset$ to $i$
Prices:
If \( i \notin W \), set \( p_i = 0 \)

\[ \alpha(i) = \min \text{ index } j \text{ s.t.} \]
\[ \begin{align*}
  & -S_i \cup S_j \neq \emptyset, \text{ and} \\
  & -S_k \cup S_j \neq \emptyset \text{ for all } k \neq j, k \neq i, k \in W
\end{align*} \]

First bidder that didn’t win because of \( i \)

If \( \alpha(i) \) does not exist, \( p_i = 0 \)

If \( \alpha(i) \) exists, \( p_i = \frac{b_{\text{key}}}{\sqrt{1 - \frac{1}{S_{\alpha(i)}}}} = b_{\alpha(i)} \sqrt{\frac{1}{S_{\alpha(i)}}} \)

Incentive Compatibility

Two key properties:

**Monotonicity:** If \( i \) wins with \( (b_i, s_i) \), wins with any \( (b_{i'}, s_{i'}) \) s.t. \( b_{i'} \geq b_i \) and \( s_{i'} \leq s_i \).

**Pr:**
Money up in ordering conflicts with no new bidders
Critical Payment: If \(i\) wins with \((b_i, s_i)\), then
\[
p_i = \min x \text{ s.t. } i \text{ would have won with } (x, s_i)
\]

\(\mathbf{Pi} \): \(x(i)\) does not exist: \(p_i = 0\)

Even if at end of ordering would win by best of \(x(i)\)

\(x(i)\) exists:

\(i\) wins iff before \(x(i)\) in ordering:

\[
\begin{array}{c}
\underbrace{1}^i \cdots \underbrace{n}_{x(i)}
\end{array}
\]

\[
\begin{align*}
&x \text{ s.t. } x = \frac{b_{x(i)}}{\sqrt{1s_{i}}} = \frac{b_{x(i)}}{\sqrt{1s_{x(i)}}} \\
&\Rightarrow x = b_{x(i)} \sqrt{\frac{1s_{i}}{1s_{x(i)}}} = p_i
\end{align*}
\]

Then: Any mechanism where losers pay 0 which has monotonicity and critical payment properties is incentive-compatible.
\(\text{\textbf{Proof: \ Fix player } i \in \{ n \}, \text{ all bids other than } i \} \) is.
\( p(b, s) \) : price charged if bids \((b, s) \)
\( u(b, s) = u(T_i) - p(b, s) \)

\textbf{Truthful bidding for nonnegative utility:}
- \( I \lor i \text{ loses: } u(u_i, T_i) \leq 0 \)
- \( I \lor i \text{ wins: } \\
  u(u_i, T_i) < u_i - p_i(u_i, T_i) = u_i - (\min x \in T_i \text{ } (x, T_i) \text{ wins} ) \text{ (critical price)} \\
  \geq u_i - u_i = 0 \quad \text{ (monotonicity)} \)

\textbf{Truthful bidding is dominant:}
\( u(T_i) : u(u_i, T_i) \geq u(b, s) \forall b, s \)
- \( I \lor (b, s) \text{ loses: } \checkmark \)
- \( I \lor T_i \neq S : u(b, s) \text{ non-negative } \checkmark \)

so \( u(T_i), (b, s) \text{ wins, } T_i \leq S \)
\( \text{Claim: } u(b, T_i) \geq u(b, s) \quad \text{then: } u(u_i, T_i) \leq u(b, s) \)
\( \text{PF: } (b, s) \text{ wins } \Rightarrow (b, T_i) \text{ wins } \)
\( \checkmark \text{ monotonicity, } T_i \leq S \)
\[ u(b, s) = u_i - p(b, s) \]
\[ u(b, T_i) = u_i - p(b, T_i) \]

\( \exists \) just need to show \( p(b, T_i) \leq p(b, s) \)

\[ p(b, s) = \min x \text{ s.t. } (x, s) \text{ wins} \text{ (critical payment)} \]

\[ (x, T_i) \text{ wins} \text{ (monotonicity)} \]

\[ \Rightarrow \ p(b, T_i) \leq x \text{ (critical payment)} \]

Claim: \( u(u_i, T_i) \geq u(b, T_i) \)

\( (b, T_i) \text{ wins} \Rightarrow \ u(b, T_i) = u_i - p(b, T_i) \)

If \( (u_i, T_i) \text{ wins}, p(b, T_i) = p(u_i, T_i) \text{ (critical payment)} \)

\[ \Rightarrow u(u_i, T_i) = u(b, T_i) \]

If \( (u_i, T_i) \text{ (tie), } u_i < p(b, T_i) \)

\[ \Rightarrow u(u_i, T_i) = 0, \]

\[ u(b, T_i) = u_i - p(b, T_i) < 0 \]
Social welfare:

Let $\text{OPT}$ be winners in welfare-maximizing allocation.

**Thm:** $\sum_{i \in \text{OPT}} v_i \leq \sqrt{m} \sum_{i \in \text{w}} v_i$

**Proof:** Let $\text{OPT}_i = \{j \in \text{OPT} : j \geq i \text{ and } T_i \cap T_j \neq \emptyset\}$

(bidders in $\text{OPT}$ later than $i$ that conflict with $i$)

$\Rightarrow \text{OPT} = \bigcup_{i \in \text{w}} \text{OPT}_i$

Let $j \in \text{OPT}_i$.

$\Rightarrow \frac{v_j}{\sqrt{|T_j|}} \leq \frac{v_i}{\sqrt{|T_i|}} \Rightarrow v_j \leq \frac{v_i}{\sqrt{|T_i|}} \sqrt{|T_j|}$

$\Rightarrow \sum_{j \in \text{OPT}_i} v_j \leq \frac{v_i}{\sqrt{|T_i|}} \sum_{j \in \text{OPT}_i} \sqrt{|T_j|}$

$\leq \frac{v_i}{\sqrt{|T_i|}} \sqrt{\text{OPT}_i} \sqrt{\sum_{j \in \text{OPT}_i} |T_j|}$

\[\sum_{i \in \text{w}} \frac{v_i}{\sqrt{|T_i|}} \sqrt{\text{OPT}_i} \sqrt{\sum_{j \in \text{OPT}_i} |T_j|}\]

Cauchy-Schwarz Inequality.
\[ | \langle x, y \rangle | \leq \| x \|_2 \cdot \| y \|_2 \]

\[ \Rightarrow \exists \sqrt{\| T \|_1} = \langle \mathbf{1}, (\sqrt{\| T \|_1})_j \rangle \]

\[ \leq \sqrt{\| T \|_1} \cdot \sqrt{\sum_{j \in \text{OPT}_i} | T_j |} \]

\[ \leq \sqrt{\sum_{j \in \text{OPT}_i} | T_j |} \]

\[ \leq \sqrt{m} \]

\[ \Rightarrow \exists \mathbf{v}_i \leq \sum_{i \in \text{OPT}_i} \mathbf{v}_i \leq \sum_{i \in \text{OPT}_i} \sqrt{m} \leq \sqrt{m} \sum_{i \in \text{OPT}_i} \mathbf{v}_i \]

\[ h \in \text{OPT}_i \iff \mathbf{v}_i \leq \mathbf{v}_i \text{ in OPT}_i \]
**Hardness:**

_Theorem:_ Unless \( P = NP \), cannot approximate social welfare to \( m^e \) for any constant \( e > 0 \)

**Proof:** Reduction from Independent Set

1. **Independent Set:**
   - **Input:** \( G = (V, E) \)
   - Find \( S \subseteq V \) s.t. if \( u, v \in S \) then \( (u, v) \notin E \), max \( |S| \)

   Known: \( NP \)-hard to give \( n^{o(1)} \)-approx for IS

Given instance \( G = (V, E) \) of IS

Create combinatorial auction with single-minded bidders:

- **bidders:** \( V \)
- **items:** \( E \)

For \( i \in V, \; \mathcal{T}_i = \{ e \in E : e \cap \{i\} \neq \emptyset \} \)

\( \mathcal{V}_i = 1 \)

**Says IS:** \( S \) can give every bidder in \( S \) their bundle

\( \Rightarrow \) social welfare \( IS \)

Says social welfare \( \alpha \): IS s.t. can all be winners, \( IS \leq \alpha \)
3. $S$ is of size $n$

3. $\text{OPT S} = \text{OPT social welfare}$

3. $\text{NP-hard to approximate social welfare to } n^{1-\varepsilon}$

3. $\text{NP-hard to approximate social welfare to } m^{\frac{1}{2}-\varepsilon}$