Combinatorial Auctions:

- Players $[n]$ — m items M
- Outcome: allocation of items to bidders, where bidder i gets $S_i \subseteq M$ and $S_i \cap S_j = \emptyset$ $\forall i \neq j$

  $\Rightarrow$ # outcomes $= (n+1)^m$
- Valuations: each i has valuation fn $v_i : 2^M \to \mathbb{R}_{\geq 0}$:

  $v_i(S)$ = value of receiving bundle S

  Assume: $v_i(\emptyset) = 0$,

  $v_i(S) \leq v_i(T) \quad \forall S \subseteq T$
- Utility = value − price
- Social welfare of $(S_1, S_2, \ldots, S_n)$ is $\sum_{i=1}^{n} v_i(S_i)$

Simple case: "Single-Minded Bidders"

Each player i has private set $T_i \subseteq M$ and private $v_i \in \mathbb{R}^+$.

$$v_i(S) = \begin{cases} v_i & \text{if } T_i \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

Bid: (b_i, S_i)

Greedy Mechanism:

Allocation Rule:

Sort and reindex so $\frac{b_1}{\sqrt{|S_1|}} \geq \frac{b_2}{\sqrt{|S_2|}} \geq \dots \geq \frac{b_n}{\sqrt{|S_n|}}$

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W = ∅
for (i = 1 to n) {
    if (S_i ∩ (⋃_{j∈W} S_j) = ∅) {
        add i to W
        give S_i to i
    }
    else give ∅ to i
}
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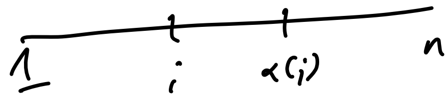
Prices:

If $i \notin W$, set $p_i = 0$

$\alpha(i)$: min index j s.t.



- $S_i \cap S_j \neq \emptyset$, and
- $S_k \cap S_j = \emptyset \quad \forall\ k < j,\ k \neq i,\ k \in W$

First bidder that didn't win because of i

If $\alpha(i)$ does not exist, $p_i = 0$

If $\alpha(i)$ exists, $p_i = \dfrac{b_{\alpha(i)}}{\sqrt{\dfrac{|S_{\alpha(i)}|}{|S_i|}}} = b_{\alpha(i)} \sqrt{\dfrac{|S_i|}{|S_{\alpha(i)}|}}$

Incentive Compatibility

Two key properties:

Monotonicity: If i wins with (b_i, S_i), wins with any (b_i', S_i') s.t. $b_i' \geq b_i$ and $S_i' \subseteq S_i$.

Pf:

Moves up in ordering, conflicts with no more bidders

Critical Payment: If i wins with (b_i, S_i), then

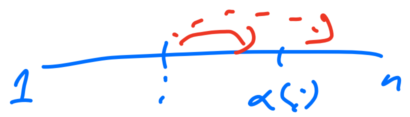
$p_i = \min x$ s.t. i would have won with (x, S_i)

Pf: $\alpha(i)$ does not exist: $p_i = 0$

Even if at end of ordering, would win by def of $\alpha(i)$

$\alpha(i)$ exists:

i wins iff before $\alpha(i)$ in ordering:



$$x \text{ s.t. } \frac{x}{\sqrt{|S_i|}} = \frac{b_{\alpha(i)}}{\sqrt{|S_{\alpha(i)}|}}$$

$$\Rightarrow x = b_{\alpha(i)} \sqrt{\frac{|S_i|}{|S_{\alpha(i)}|}} = p_i$$

Thm: Any mechanism where losers pay 0 which has monotonicity and critical payment properties is incentive-compatible.

Pf: Fix player $i \in [n]$, all bids other than i's.

$p(b, S)$: price charged i if bids (b, S)

$u(b, S) = v(T_i) - p(b, S)$

<u>Truthful bidding $\Rightarrow$ nonnegative utility:</u>

If i loses: $u(v_i, T_i) = 0$

If i wins:

$u(v_i, T_i) = v_i - p_i(v_i, T_i) = v_i - (\min x \text{ s.t. } (x, T_i) \text{ wins})$ (critical price)

$\geq v_i - v_i = 0$ (monotonicity)

<u>Truthful bidding is dominant:</u>

<u>WTS:</u> $u(v_i, T_i) \geq u(b, S) \quad \forall b, S$

If (b, S) loses: ✓

If $T_i \not\subseteq S$: $u(b, S)$ nonpositive ✓

So WLOG, (b, S) wins, $T_i \subseteq S$

<u>Claim:</u> $u(b, T_i) \geq u(b, S)$ then: $u(v_i, T_i) \geq u(b, T_i)$

<u>Pf:</u> (b, S) wins $\Rightarrow$ (b, T_i) wins

monotonicity, $T_i \subseteq S$

$\Rightarrow u(b, S) = v_i - p(b, S)$

$u(b, T_i) = v_i - p(b, T_i)$

$\Rightarrow$ just need to show $p(b, T_i) \leq p(b, S)$

$p(b, S) = \min x$ s.t. (x, S) wins (critical payment)

$\Rightarrow (x, T_i)$ wins (monotonicity)

$\Rightarrow p(b, T_i) \leq x$ (critical payment)

<u>Claim</u>: $u(v_i, T_i) \geq u(b, T_i)$

(b, T_i) wins $\Rightarrow u(b, T_i) = v_i - p(b, T_i)$

If (v_i, T_i) wins, $p(b, T_i) = p(v_i, T_i)$ (critical payment)

$\Rightarrow u(v_i, T_i) = u(b, T_i)$

If (v_i, T_i) loses, $v_i \underset{\uparrow}{<} p(b, T_i)$

critical payment + monotonicity

$\Rightarrow u(v_i, T_i) = 0$,

$u(b, T_i) = v_i - p(b, T_i) < 0$

Social welfare:

Let OPT be winners in welfare-maximizing allocation

Thm: $\sum_{i \in OPT} v_i \leq \sqrt{m} \sum_{i \in W} v_i$

Pf: Let $OPT_i = \{ j \in OPT : j \geq i \text{ and } T_i \cap T_j \neq \emptyset \}$

(bidders in OPT later than i that conflict with i)

$\Rightarrow OPT = \bigcup_{i \in W} OPT_i$

Let $j \in OPT_i$

$$\Rightarrow \frac{v_j}{\sqrt{|T_j|}} \leq \frac{v_i}{\sqrt{|T_i|}} \Rightarrow v_j \leq \frac{v_i}{\sqrt{|T_i|}} \sqrt{|T_j|}$$

$$\Rightarrow \sum_{j \in OPT_i} v_j \leq \frac{v_i}{\sqrt{|T_i|}} \sum_{j \in OPT_i} \sqrt{|T_j|}$$

$$\leq \frac{v_i}{\sqrt{|T_i|}} \sqrt{|OPT_i|} \sqrt{\sum_{j \in OPT_i} |T_j|}$$

Cauchy-Schwarz Inequality:

$$|\langle x, y\rangle| \le \|x\|_2 \cdot \|y\|_2$$

$$\Rightarrow \sum_{j \in OPT_i} \sqrt{|T_j|} = \langle \vec{1}, (\sqrt{|T_j|})_{j \in OPT_i} \rangle$$

$$\le \sqrt{|OPT_i|} \sqrt{\sum_{j \in OPT_i} |T_j|}$$

$$\le v_i \sqrt{\sum_{j \in OPT_i} |T_j|}$$

$|OPT_i| \le |T_i|$: every $j \in OPT_i$ conflicts with i on different item in T_i, since all in OPT $\Rightarrow$ disjoint

$$\le v_i \sqrt{m}$$

$$\Rightarrow \sum_{i \in OPT} v_i \le \sum_{i \in W} \sum_{j \in OPT_i} v_j \le \sum_{i \in W} v_i \sqrt{m} = \sqrt{m} \sum_{i \in W} v_i$$

b/c $OPT = \bigcup_{i \in W} OPT_i$

Hardness:

Thm: Unless $P=NP$, cannot approximate social welfare to $m^{\frac{1}{2}-\varepsilon}$ for any constant $\varepsilon>0$

Pf: Reduction from Independent Set

Independent Set:

- input $G=(V,E)$
- Find $S \subseteq V$ s.t. $\underbrace{\text{if } u,v \in S \text{ then } \{u,v\} \notin E}_{\text{Indep. Set}}$, max $|S|$

known: NP-hard to give $n^{1-\varepsilon}$-approx for IS

Given instance $G=(V,E)$ of IS

Create combinatorial auction with single-minded bidders:

bidders $=V$ items $=E$

For $i \in V$, $T_i = \{e \in E : e \cap \{i\} \neq \emptyset\}$

$v_i = 1$

Sps I.S. S: can give every bidder in S their bundle

$\Rightarrow$ social welfare $|S|$

Sps social welfare α: $\exists S$ s.t. can all be winners, $|S|=\alpha$

$\Rightarrow$ S IS of size α

$\Rightarrow$ OPT IS = OPT social welfare

$\Rightarrow$ NP-hard to approximate social welfare to $n^{1-\varepsilon}$

$\Rightarrow$ NP-hard to approximate social welfare to $m^{\frac{1}{2}-\varepsilon}$