4/7/22: "General" Mechanism Design (not just single parameter)

Setup:
- Bidders (players $[n]$
- Finite set Ω of outcomes (allocations)
- Each bidder i has valuation function $v_i : \Omega \to \mathbb{R}_{\geq 0}$ (private)
- Social Welfare of $w \in \Omega$ is $\sum_{i=1}^{n} v_i(w)$
- Utility of player i = value − price

Ex: Single item: Ω is n+1 possible allocations

$$v_i(w) = \begin{cases} v_i & \text{if } i \text{ wins in } w \\ 0 & \text{otherwise} \end{cases}$$

Single Parameter Environments:

$$\Omega = X \qquad v_i(w) = v_i w_i$$

Combinatorial Auctions:

- Players $[n]$ — m items M
- Player i has valuation for each "bundle" $S \subseteq M$

$$V_i : 2^M \to \mathbb{R}_{\geq 0}$$

- Ω = all possible allocations

$$f : M \to [n]$$

Thm [Vickrey-Clarke-Groves]: In every general mechanism design environment, there is an incentive-compatible mechanism which maximizes social welfare.

Issues/Notes:

- Computationally inefficient
- Myerson doesn't apply, and what does "monotonicity" mean?

  $\Rightarrow$ have to also design prices
- What is a bid?

  Send bid function $b_i : \Omega \to \mathbb{R}_{\geq 0}$

  IC if $b_i = v_i$ is dominant strategy

VCG Mechanism:

Outcome Rule:

$$x(b) = \underset{\omega \in \Omega}{\arg\max} \sum_{i=1}^{n} b_i(\omega)$$

Pricing Rule:

Intuition from Vickrey (second-price) for single item

How does player i participating affect social welfare of everyone else?

Let x be allocation when i participates,
x' " " i does not participate

i does not get item: (in x)

$$\sum_{j \neq i} v_j x_j = \max_{j \in [n]} v_j = \sum_{j \neq i} v_j x'_j$$

(i has no affect on social welfare)

i does get item: (in x)

$$\Rightarrow \sum_{j \neq i} v_j x_j = 0$$

$$\sum_{j \neq i} v_j x'_j = \max_{j \neq i} v_j$$

$$\Rightarrow \text{damage} = \max_{j \neq i} v_j - 0 = \max_{j \neq i} v_j$$

= price!

So think of price as "harm caused to other players" (externality)

In general:

Given bids b, let $\omega^* = \underset{\omega \in \Omega}{\operatorname{argmax}} \sum_{i=1}^{n} b_i(\omega)$

$\Rightarrow$ Externality of i:

$$\underbrace{\max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega)}_{\text{without } i} - \underbrace{\sum_{j \neq i} b_j(\omega^*)}_{\text{with } i} = p_i(b)$$

VCG Mechanism: max social welfare, charge externality

Analysis: Fix $i \in [n]$, other bids b_{-i}

If i bids b_i, outcome $\omega^* = \underset{\omega \in \Omega}{\operatorname{argmax}} \sum_{j=1}^{n} b_j(\omega)$

$b_i = v_i$ is dominant strategy:

Utility of player i is

$$v_i(\omega^*) - p_i(b) = v_i(\omega^*) - \left(\max_{\omega \in \Omega} \sum_{j \neq i} b_j(\omega) - \sum_{j \neq i} b_j(\omega^*)\right)$$

$$= \underbrace{v_i(w^*) + \sum_{j\neq i} b_j(w^*)}_{b_i \text{ affects } w^*} - \underbrace{\max_{w\in\Omega} \sum_{j\neq i} b_j(w)}_{\text{independent of } b_i}$$

$\Rightarrow$ to max utility, player i bids to maximize

$$v_i(w^*) + \sum_{j\neq i} b_j(w^*)$$

If bid b_i, w^* will maximize $\sum_{j=1}^n b_j(w^*) = v_i(w^*) + \sum_{j\neq i} b_j(w^*)$

$\Rightarrow$ if $b_i = v_i$, maximize utility!

$p_i(b) \geq 0$:

$$\max_{w\in\Omega} \sum_{j\neq i} b_j(w) - \sum_{j\neq i} b_j(w^*) \geq 0$$

Nonnegative utility for truthful bidders:

Let $w' = \arg\max_{w\in\Omega} \sum_{j\neq i} b_j(w)$

If $b_i = v_i$, utility of player i =

$$= v_i(w^*) + \sum_{j\neq i} b_j(w^*) - \sum_{j\neq i} b_j(w')$$

$$= \sum_{j=1}^n b_j(w^*) - \sum_{j\neq i} b_j(w')$$

$$\geq \sum_{j=1}^{n} b_j(w') - \sum_{j\neq i} b_j(w') \qquad (\text{def of } w^*)$$

$$= b_i(w') = v_i(w') \geq 0$$

"Practical" Issues with VCG:

- Computing outcome is computationally hard
- Preference elicitation / size of bid
- Bad for revenue:

Two items A, B

Two players: $v_1(AB) = 1$, 0 otherwise

$v_2(AB) = v_2(A) = 1$, 0 otherwise

max welfare $= 1$, price $= 1$:

both to 1, $\underbrace{\text{both to 2, B to 1 A to 2}}_{\text{player 2 charged 1}}$

player 1 charged 1

Add new player 3:

$v_3(AB) = v_3(B) = 1$, 0 otherwise

max welfare $= 2$: A to 2, B to 3

$p_2 = 1 - 1 = 0$

welfare w/out 2 participating

welfare of all other players in chosen outcome

$p_3 = 1 - 1 = 0$

No revenue!