20.1 Introduction

Today we’re finally going to go beyond single-parameter environments. We will define and analyze the main result from classical mechanism design, known as the VCG mechanism.

20.2 Setup & Examples

20.2.1 General Setup

- Bidders/agents/players \([n]\)
- Finite set \(\Omega\) of outcomes (think allocations)
- Each player \(i\) has a valuation function \(v_i : \Omega \rightarrow \mathbb{R}_{\geq 0}\)
- The social welfare of some outcome \(w \in \Omega\) is \(\sum_{i=1}^{n} v_i(w)\).

20.2.2 Examples

**Single Item:** \(\Omega\) is the \(n + 1\) possible allocations (\(n\) different winners or no winner), and \(v_i(w)\) is equal to \(v_i\) if \(i\) wins the item in \(w\) and 0 otherwise. Note that in the more general setting, we can have more interesting valuation functions even for a single item. Maybe player \(i\) would like to get the item, but if they don’t then they would still prefer player \(j\) to get the item over player \(k\). We can model that now by an appropriate valuation function \(v_i\)!

**Single-Parameter Environments:** \(\Omega = X\) (the outcomes are the feasible allocations), and \(v_i(w) = v_iw_i\) (here \(w\) is an \(n\)-dimensional vector since \(\Omega = X\)).

**Combinatorial Auctions:** We’ll talk a lot more about these next lecture. But they’re intuitive and good to keep in mind for general auctions, so I’ll give the definition here.

- Players \([n]\).
- \(m\) items \(M\).
- Each player \(i\) has a valuation for each “bundle” \(S \subseteq M\), so \(v_i : 2^M \rightarrow \mathbb{R}\) (note that there are \(2^m\) private parameters now instead of just one!)
- The outcomes are allocations \(f : M \rightarrow [n]\), so \(v_i(f) = v_i(\{j : f(j) = i\})\).
20.3 Main Result

Today we’re going to try to prove the following result:

**Theorem 20.3.1** (Vickrey-Clarke-Groves). *In every general mechanism design environment, there is an incentive-compatible mechanism which maximizes social welfare.*

Clearly this is a very powerful theorem! But there are a few subtleties and caveats that we should mention before we actually prove it. First, this is not an “awesome” auction: we cannot compute it in polynomial time. This should not be surprising, since we cannot efficiently compute the welfare-maximizing allocation even in single-parameter environments like knapsack auctions.

Second, for the first time in a couple weeks we are not able to use Myerson to get prices once we’ve decided how to choose an outcome (since we’re no longer in a single-parameter environment). So we’ll have to argue incentive compatibility directly.

Third, we have to be a bit careful about what we mean by a “bid” in this context. The obvious thing, which is what we’re going to do, is to let a “bid” consist of all the private information, i.e., bidder $i$ sends a bid function $b_i : \Omega \rightarrow [0, \infty)$. Then a mechanism is incentive compatible if truthfully sending $b_i = v_i$ is a dominant strategy for player $i$. This clearly has some issues in practice, since in general $v_i$ can be huge, and there’s a lot of interesting work on preference elicitation, bidding languages, and other extensions of the setting to enable bids to be a little more reasonable. But that’s what we’ll use for today.

20.3.1 The VCG Mechanism

To define the mechanism we need to define the allocation/outcome rule and the prices to charge. Like always, we’ll assume that the utility of a player is its valuation minus the price it pays. Like in the single-parameter environment, we’ll try to use a two-stage design process: first figure out the outcome rule, and then figure out a pricing rule to make this incentive compatible.

**Outcome rule:** We don’t really have much choice here. We need to maximize social welfare, so we’re basically forced to use the outcome rule which does this. Formally, given bids $b$ (where $b_i : \Omega \rightarrow [0, \infty)$), we’ll choose the outcome

$$x(b) = \arg \max_{w \in \Omega} \sum_{i=1}^{n} b_i(w)$$

**Pricing rule:** Now comes the tricky part: how should we design prices to make this incentive compatible (and is that even possible)?

- In the single-parameter setting, we proved (Myerson’s Lemma) that such a pricing function exists if the allocation rule is monotone, and then proved that the welfare-maximizing allocation rule is monotone. But what does monotone even mean here? What if there’s one parameter where we overbid and one parameter where we underbid – should we get more or less value from this?
In the single-item setting, we showed that the right price was the “critical bid”. But now even for a single-item, players are allowed to have pretty complicated valuations. What does the “critical bid” mean here?

Let’s go back to the Vickrey (second-price) auction for a single-item to get some inspiration. We’re just going to have to look at it in a slightly different way. Consider some player \( i \). How much does player \( i \) hurt everyone else by participating in the auction?

- If \( i \) does not get the item, then it doesn’t hurt anyone – the only difference between it participating and not participating is that the winner might be charged a higher price, but the social welfare is unchanged. So the social welfare of everyone else is also unchanged.

- If \( i \) does get the item, then it was the highest bid. So everyone else has social welfare 0, i.e., \( \sum_{j \neq i} v_j x_j = 0 \). But if \( i \) did not participate, then the item would have gone to the bidder with the highest valuation other than \( i \), i.e., the second-highest bid. So in that case, we would have that \( \sum_{j \neq i} v_j x_j = \arg \max_{j \neq i} v_j \). So player \( i \), by participating in the auction, made the combined social welfare of all other players go from \( \arg \max_{j \neq i} v_j \) to 0. So the “total harm” caused by player \( i \) is the second highest bid, which is the price that player \( i \) was charged!

We’re going to generalize this idea to the general mechanism design setting: we’ll charge player \( i \) the eternity they cause. Formally, given bids \( b \), let \( w^* \) be the welfare-maximizing outcome. Then the price we’ll charge player \( i \) is

\[
p_i(b) = \max_{w \in \Omega} \sum_{j \neq i} b_j(w) - \sum_{j \neq i} b_j(w^*).
\]

Note that this is precisely the outcome maximizing social welfare without player \( i \) minus the social welfare of the other players when player \( i \) participates, so it is the harm that player \( i \) causes the other players.

This pricing rule, combined with the social welfare-maximizing outcome rule, is the VCG Mechanism.

20.3.2 Analysis

Clearly VCG maximizes social welfare (by definition), so we just need to prove that it is incentive compatible, i.e., bidding \( b_i = v_i \) is a dominant strategy for all players \( i \). So fix some player \( i \in [n] \) and other bids \( b_{-i} \).

The utility of player \( i \) under bids \( b \) is \( v_i(w^*) - p_i(b) \), where \( w^* = \arg \max_{w \in \Omega} \sum_{j=1}^{n} b_j(w) \). Plugging in the payment rule from the mechanism, we get that the utility of player \( i \) is

\[
v_i(w^*) - p_i(b) = v_i(w^*) - \max_{w \in \Omega} \sum_{j=1}^{n} b_j(w) + \sum_{j \neq i} b_j(w^*)
\]

\[
= \left(v_i(w^*) + \sum_{j \neq i} b_j(w^*)\right) - \max_{w \in \Omega} \sum_{j \neq i} b_j(w)
\]

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The last term here is independent of player $i$’s bid $b_i$: nothing $i$ does can affect this term. So player $i$ is trying to maximize $v_i(w^*) + \sum_{j \neq i} b_j(w^*)$, which it can affect since $b_i$ will affect $w^*$. What is this sum? If $b_i = v_i$ then it is $\sum_{j \in [n]} b_j(w^*)$, which is exactly what the outcome rule is trying to maximize! So if player $i$ bids $v_i$, then the outcome rule is choosing the outcome that is best for player $i$. If player $i$ bids some $b_i \neq v_i$, then the outcome rule chooses some $w^*$ that maximizes social welfare for $b_i$ rather than for $v_i$, which is clearly worse for player $i$. Thus bidding $b_i$ is a dominant strategy.

To see this intuitively, note that what the VCG mechanism is doing is “aligning incentives”: because of the price chosen, in order to maximize their own utility player $i$ wants to maximize total social welfare. Since that’s what the mechanism is trying to do, the best thing for player $i$ to do is be truthful and let the mechanism do this.

So we’re almost done. We just need to prove that utilities are never negative for a truthful bidder and that prices are never negative. Clearly $p_i(b)$ is never negative, since the $w^*$ is one particular instantiation of the max. To prove nonnegative utilities, let $w' = \arg\max_{w \in \Omega} \sum_{j \neq i} b_j(w)$. Then the price is

$$p_i(b) = \sum_{j \neq i} b_j(w') - \sum_{j \neq i} b_j(w^*)$$

$$= b_i(w^*) - \left( \sum_{j=1}^n b_j(w^*) - \sum_{j \neq i} b_j(w') \right).$$

The first term here is what player $i$ bid, so for a truthful bidder this will be $v_i(w^*)$. Thus if $i$ is truthful, it will have utility

$$v_i(w^*) - p_i(b) = \sum_{j=1}^n b_j(w^*) - \sum_{j \neq i} b_j(w') \geq \sum_{j=1}^n b_j(w') - \sum_{j \neq i} b_j(w') = b_i(w') \geq 0$$