1/27/22: 2-player zero-sum games

Notes: - HW 1 released, due in 2 weeks (beginning of lecture)

- Recording not great

Def [One-shot simultaneous-move game]:

- n players $[n] = \{1, 2, \dots, n\}$
- Finite set S_i of strategies/actions for player i

  $S = S_1 \times S_2 \times \dots \times S_n$ set of strategy profiles
- Utility function $u_i : S \to \mathbb{R}$ for each $i \in [n]$

Notation: $\Delta_k = \{ x \in \mathbb{R}^k : x_i \geq 0 \ \forall i \in [k], \ \sum_{i=1}^{k} x_i = 1 \}$

set of probability vectors/distributions over $[k]$

Def [Nash equilibrium]: Let $\mu_i \in \Delta_{S_i}$ $\forall i \in [n]$. Then $(\mu_1, \mu_2, \dots, \mu_n)$ is a Nash Equilibrium if $\forall i \in [n]$ and $\mu_i' \in \Delta_{S_i}$

$$\mathop{E}_{s_j \sim \mu_j \ \forall j \in [n]} \left[u_i(s_1, s_2, \dots, s_n) \right] \geq \mathop{E}_{\substack{s_i \sim \mu_i' \\ s_j \sim \mu_j \ \forall j \neq i}} \left[u_i(s_1, s_2, \dots, s_n) \right]$$

Notes:

- A single strategy $s \in S_i$, or $\mu \in \Delta_{S_i}$ with exactly one nonzero entry, is a pure strategy
- $\mu \in \Delta_{S_i}$ called a mixed strategy
- If all μ_i's are pure strategies, called a pure Nash equilibrium
- Sometimes call a NE a mixed NE to distinguish from pure

<u>Thm</u> (Nash): Every game with a finite # players, finite # pure strategies, has at least one NE

Possibly weird things to keep in mind:

- implicitly assuming every player knows mixed strategy of every other player, but not action drawn
- "stable", but one-shot

Can simplify definition: only need to consider deviations to pure strategies!

Thm: If $(\mu_1, \mu_2, \ldots, \mu_n)$ is not a NE, then $\exists\, i \in [n],\ a \in S_i$ s.t.

$$\mathop{E}_{s_j \sim \mu_j\ \forall j \in [n]} \left[u_i(s_1, s_2, \ldots, s_n) \right] < \mathop{E}_{\substack{s_j \sim \mu_j\ \forall j \neq i \\ s_i = a}} \left[u_i(s_1, s_2, \ldots, s_n) \right]$$

Pf sketch: Since not a NE, $\exists\, i \in [n],\ \mu_i' \in \Delta_{S_i}$ s.t. i has incentive to deviate to μ_i' from μ_i

But μ_i' convex combination of pure strategies!

$\Rightarrow \mu_i'$ no better than best pure strategy

Lots of generalizations and other definitions!

- subgame perfect Nash
- Bayes-Nash
- strong Nash

$\vdots$

Today: $n=2$, "zero-sum"

Next class: $n=2$, non-zero-sum

Future: $n>2$

Notation: Bimatrix game

- 2 players - $S_1=[N]$ - $S_2=[M]$
- Matrices $A, B \in \mathbb{R}^{N \times M}$
- $u_1(i,j)=A_{ij}$ - $u_2(i,j)=B_{ij}$

useful notation b/c linear algebra!

Claim: Let $x \in \Delta_N$ (mixed strategy for player 1

$y \in \Delta_M$ (mixed strategy for player 2

Then $\mathop{E}_{\substack{i\sim x\\ j\sim y}}[u_1(i,j)] = x^T A y$, and

$\mathop{E}_{\substack{i\sim x\\ j\sim y}}[u_2(i,j)] = x^T B y$

Pf: $x^T A y = \sum_{i=1}^{N} \sum_{j=1}^{M} A_{ij} x_i y_j$ (LA)

$= \sum_{i=1}^{N} \sum_{j=1}^{M} u_1(i,j) \cdot \Pr[\text{player 1 plays } i] \cdot \Pr[\text{player 2 plays } j]$ (def)

$$= \mathop{E}_{\substack{i \sim x \\ j \sim y}} [u_1(i,j)]$$ ✓

Def: A bimatrix game is zero-sum if

$B_{ij} = -A_{ij} \quad \forall i \in [N], j \in [M]$

$\Rightarrow$ in zero-sum game, just need single matrix A:

player 1 chooses x to maximize $x^T A y$,

player 2 chooses y to minimize $x^T A y$

Thm [von Neumann]: In a two-player zero-sum game, we can compute a Nash equilibrium in polynomial time

Note: Not original formulation!

Sps row player uses mixed strategy $x \in \Delta_N$.

$\Rightarrow x^T A \in \mathbb{R}^M$, and $(x^T A)_j$ is expected utility for row player if column player uses pure strategy j

So what does column player do?

Choose j minimizing $(x^T A)_j$!

$\Rightarrow$ if row player uses x, will end up getting

$$\min_{j \in [M]} (x^T A)_j \quad \text{utility}$$

So what will row player do?

Choose $x \in \Delta_N$ maximizing $\min_{j \in [M]} (x^T A)_j$

Can we compute this?

$$
\begin{aligned}
v_r = \max \quad & v \\
\text{s.t.} \quad & x_i \geq 0 \qquad \forall i \in [N] \\
& \sum_{i=1}^{N} x_i = 1 \\
& (x^T A)_j \geq v \quad \forall j \in [M]
\end{aligned}
$$

$$(x^T A)_j = \sum_{i=1}^{N} A_{ij} x_i$$

Linear programming!

Let $p \in \Delta_N$ be optimal solution to LP

Thm: Let (p^*, q^*) be a NE, and let $v^* = p^{*T} A q^*$ be expected utility of row player. Then $v_r = v^*$.

Pf: $v_r \le v^*$: Sps $v_r > v^*$. Then if row player switches to p, gets utility

$$p^T A q^* \ge \min_{j \in [M]} (p^T A)_j = v_r > v^*$$

$\Rightarrow$ would have incentive to deviate

$v^* \le v_r$: Since column player doesn't want to deviate,

$$v^* = \min_{j \in [M]} (p^{*T} A)_j$$

$$\le v_r,$$ since v^* feasible solution for LP, v_r optimal for LP

Now similar thought process for column player:

If column player plays $y \in \Delta_M$, row player chooses $i \in [N]$ maximizing $(Ay)_i$

$\Rightarrow$ column player plays $y \in \Delta_M$ minimizing $\max_{i \in [N]} (Ay)_i$

compute via LP:

$$
\begin{aligned}
v_c = \min \quad & v \\
\text{s.t.} \quad & y_j \geq 0 \qquad j \in [m] \\
& \sum_{j=1}^{m} y_j = 1 \\
& (Ay)_i \leq v \qquad \forall i \in [N]
\end{aligned}
$$

$$(Ay)_i = \sum_{j=1}^{m} A_{ij} y_j$$

Analogous to before: $v_c = v^*$

Let $p \in \Delta_N$ opt solution to row LP,

$q \in \Delta_m$ opt solution to column LP

<u>Thm</u>: (p, q) is a Nash equilibrium

<u>Pf</u>: $p^T A q \geq v_r = v^*$ since p opt sol to row LP

$p^T A q \leq v_c = v^*$ since q opt sol to col LP

$\Rightarrow p^T A q = v^*$

Sps row player deviates to pure strategy $k \in [N]$:

row player gets utility

$$(Aq)_k \leq \max_{i \in [N]} (Aq)_i = v_c = v^*$$

Sps col player deviates to pure strategy $k \in [M]$:

has cost

$$(p^T A)_k \geq \min_{j \in [M]} (p^T A)_j = v_r = v^*$$

$\Rightarrow$ Nash!