4/5/22: Simple Near-Optimal Auctions

Last time: single-parameter environment where each valuation v_i drawn from regular distribution F_i

Today: all distributions still regular

virtual valuation: $\varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$

Thm: E[revenue] = E[virtual welfare]

$$E\left[\sum_{i=1}^{n} p_i(v)\right] = E\left[\sum_{i=1}^{n} \varphi_i(v_i)\, x_i(v)\right]$$

Single-item, $F_i = F$ $\forall i$: Vickrey (second-price) with reserve $\varphi^{-1}(0)$

Prior-Free Auction: what if we don't know F?

Let OPT_F be optimal auction for F

(maximizes expected revenue among all IC auctions)

Thm: $$\underset{v_1,\dots,v_{n+1}\sim F}{E}\left[\text{Revenue}(\text{Vickrey on } v_1, v_2, \dots, v_{n+1})\right] \geq \underset{v_1,\dots,v_n\sim F}{E}\left[\text{Revenue}(OPT_F \text{ on } v_1,\dots,v_n)\right]$$

(Bulow-Klemperer)

Interpretation: instead of learning F, get another bidder!

Pf: "Fake auction" A with $n+1$ bidders

- OPT_F on $[n]$
- if item not sold, give to bidder $n+1$ for free

Properties of A:

1) Incentive-Compatible:
   - OPT_F IC on $[n]$
   - bidder $n+1$ does not affect anything
2) E[Revenue of A on $n+1$ bidders] = E[Revenue of OPT_F on n bidders]
   - runs OPT_F on $[n]$, no revenue from $n+1$
3) A always sells item

By (2), just want to show that

E[Revenue of Vickrey on $n+1$ bidders] $\geq$ E[Revenue of A]

Claim: Vickrey maximizes revenue among all IC auctions that always sell item.

$\Rightarrow$ at least as much revenue as A

OPT: max $E[\sum_{i=1}^{n} \varphi_i(v_i) x_i]$ over all allocation rules s.t. $\sum_{i=1}^{n} x_i = 1$

$\Rightarrow$ best thing is to give to highest bidder (since F regular)

$\Rightarrow$ Vickrey!

Single item, different F_i

OPT auction: Give to bidder with highest virtual valuation if > 0

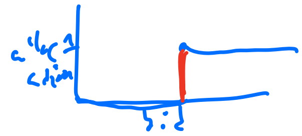
maximizes $E[\sum_{i=1}^{n} \varphi_i(v_i) x_i] = E[\sum_{i=1}^{n} p_i(v)]$

Two issues that make this "complicated":

Issue 1: Winner is hard to "explain"

- Not necessarily the highest bidder!

<u>Issue 2</u>: Price is complicated!



"critical bid": bid necessary for winner to have won

$$\max\left(\varphi_i^{-1}(0),\ \varphi_i^{-1}\left(\max_{j \neq i} \varphi_j(v_j)\right)\right)$$

- Different reserve price for each bidder
- "second-highest" very complicated since virtual

<u>Simpler Auction</u>:

Vickrey with bidder-specific reserve

- Sell to highest bidder above their reserve!

<u>Notation</u>: $z^+ = \max(z, 0) \quad \forall z \in \mathbb{R}$

<u>Let</u>: $t \in \mathbb{R}_{\geq 0}$:

$$\Pr\left[\max_{i \in [n]} \varphi_i(v_i)^+ \geq t\right] = \frac{1}{2}$$

(if t does not exist, e.g. F_i not continuous, standard workarounds: Exercise 6.2 of Roughgarden)

Set $r_i = \varphi_i^{-1}(t) \quad \forall i \in [n]$

__Auction:__ Give item to highest bidder that meets their reserve:

$$\operatorname*{argmax}_{i:\, b_i \ge r_i} b_i$$

__Price:__ $\max\left(r_i, \max_{j \ne i:\, b_j \ge r_j} b_j\right)$ (from Myerson)

__Thm:__ expected revenue $\ge \frac{1}{2} \cdot$ expected revenue of optimal auction

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Prophet Inequality:

__Setup:__ Distributions $G_1, \dots, G_n$ (known)

- At time j, "prize" $\pi_j \sim G_j$.

  Decide: accept π_j, end process

  __or__ reject π_j, go to time $j+1$

Goal: maximize prize accepted.

Prophet knows $\pi_1, \dots, \pi_n$

want strategy competitive to $E\left[\max_{i \in [n]} \pi_i\right]$

Alg: threshold t, accept π_i iff $\pi_i \geq t$

choose t s.t. $\Pr[\max_{i \in [n]} \pi_i \geq t] = \frac{1}{2}$

<u>Thm</u>: $E[\text{reward of Alg}] \geq \frac{1}{2} E[\max_{i \in [n]} \pi_i]$

<u>Pf</u>: w.p. $\frac{1}{2}$: no reward (def of t)

w.p. $\frac{1}{2}$: get $\geq t$

If π_i <u>only</u> prize above t, get <u>additional</u> $\pi_i - t$

$\cdot \Pr[\pi_i$ only prize is above $t)$

$$E(\text{reward}) \geq \frac{1}{2} \cdot t + \sum_{i=1}^{n} E[\pi_i - t \mid \pi_i \text{ only prize above } t]$$

$$= \frac{1}{2} t + \sum_{i=1}^{n} \underbrace{E[\pi_i - t \mid \pi_i \geq t] \cdot \Pr[\pi_i \geq t]}_{E[(\pi_i - t)^+]} \cdot \underbrace{\Pr[\pi_j < t \ \forall j \neq i]}_{\geq \frac{1}{2}}$$

$$= \frac{1}{2} t + \frac{1}{2} \sum_{i=1}^{n} E[(\pi_i - t)^+]$$

$$= \frac{1}{2} \left(t + \sum_{i=1}^{n} E[(\pi_i - t)^+] \right)$$

$$\mathbb{E}[\max_{i \in [n]} \pi_i] = \mathbb{E}[t + \max_{i \in [n]} (\pi_i - t)]$$

$$= t + \mathbb{E}[\max_{i \in [n]} (\pi_i - t)]$$

$$\leq t + \mathbb{E}[\max_{i \in [n]} (\pi_i - t)^+]$$

$$\leq t + \mathbb{E}[\sum_{i=1}^{n} (\pi_i - t)^+]$$

$$\leq t + \sum_{i=1}^{n} \mathbb{E}[(\pi_i - t)^+]$$

Use prophet inequality to analyze our auction

Thm: expected revenue $\geq \frac{1}{2} \cdot$ expected revenue of optimal auction

Pf sketch:

Think of $\varphi_i(v_i)^+$ as prize π_i

$\Rightarrow t$ in auction same as in prophet inequality!

Difference:

- auction takes largest bid with prize above t
- Prophet inequality takes first bid with prize above t

But proof of prophet inequality works even if we take arbitrary prize over threshold!

- Only gave ourselves $>t$ if exactly one prize above t

$\Rightarrow$ if x is allocation vector of our mechanism:

$$E\Big[\sum_{i=1}^{n} \varphi_i(v_i)^+ x_i\Big] \geq \frac{1}{2} E\Big[\max_{i \in [n]} \varphi_i(v_i)^+\Big] \quad \text{(prophet)}$$

we only sell to i if $v_i \geq r_i = \varphi_i^{-1}(t)$

$\Rightarrow \varphi_i(v_i) \geq t \geq 0$

$\Rightarrow \varphi_i(v_i)^+ = \varphi_i(v_i)$

$\Rightarrow$ if $x_i = 1$, then $\varphi_i(v_i)^+ = \varphi_i(v_i)$

$$\Rightarrow E\Big[\sum_{i=1}^{n} \varphi_i(v_i)^+ x_i\Big] = E\Big[\sum_{i=1}^{n} \varphi_i(v_i) x_i\Big]$$

$$\Rightarrow E\Big[\sum_{i=1}^{n} \varphi_i(v_i) x_i\Big] \geq \frac{1}{2} E\Big[\max_{i \in [n]} \varphi_i(v_i)^+\Big]$$

$\Rightarrow$ expected revenue $\geq \frac{1}{2} \cdot$ expected revenue of optimal auction