3/31/22: Revenue-Maximizing Auctions

Note: start thinking about course projects!

Still want incentive compatibility, but want to maximize revenue rather than social surplus

Trivial example: single-item, single-bidder

"posted price mechanism"

-Need to make some extra assumptions!

Setup:

- Single-Parameter environment: bidders $[n]$, feasible set $X \subseteq \mathbb{R}^n_{\geq 0}$

known to mechanism

- For each $i \in [n]$, distribution F_i
  - support $(F_i) \subseteq [0, v_{max}]$
  - Also use F_i to denote CDF (cumulative distribution function):

$$F_i(z) = \Pr_{x \sim F_i}(x \leq z) \qquad F_i(v_{max}) = 1$$

  - f_i is probability density function:

$$\int_0^z f_i(x)\,dx = F_i(z)$$

- private $v_i \sim F_i$

- Goal: IC mechanism maximizing expected revenue

Examples:

single-item, single-bidder:

set price $r \Rightarrow E[\text{revenue}] = r\,(1 - F(r))$

revenue if sale

Prob. of sale

$F_i = \text{Unif}(0,1)$: set $r = \frac{1}{2}$, $E[\text{revenue}] = \frac{1}{4}$

$F(x) = x$

Single-item, Two bidders:

Sps F_1, F_2 uniform $(0,1)$

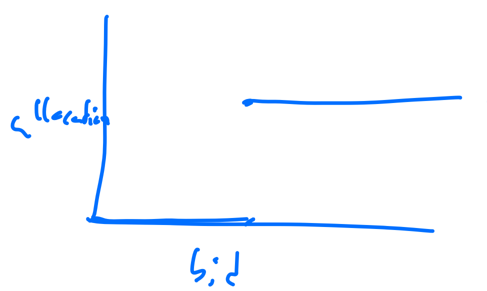
Second-price (Vickrey) auction:

$E[\text{revenue}] = E[\text{second-highest valuation}]$

$= 1/3$

"Reserve price r" auction: Give item to highest bidder *if* highest bid $\geq r$.

Otherwise, no one gets item.



Price from Myerson:

$\max(r, \text{second-highest})$

$r = \frac{1}{2}$:

$$E[\text{revenue}] = \Pr[\text{both above } \tfrac{1}{2}] \cdot E[\text{second highest} \mid \text{both above } \tfrac{1}{2}] + \Pr[\text{highest} \geq \tfrac{1}{2}, \text{second} < \tfrac{1}{2}] \cdot \frac{1}{2}$$

$$= \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{12} > \frac{1}{3}$$

Back to main setting:

Single-parameter environment $\Rightarrow$ Myerson applies

$\Rightarrow$ for allocation function $x: [0, v_{max}]^n \to X$, prices p from Myerson

$\Rightarrow$ Revenue maximizing auction is monotone x, p from Myerson maximizing

$$\mathop{E}_{v \sim F_1 \times F_2 \times \dots \times F_n} \left[\sum_{i=1}^{n} p_i(v) \right]$$

Def: The virtual valuation of bidder i with valuation v_i is

$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Ex: F_i uniform$(0,1)$: $F_i(v_i) = v_i$, $f_i(v_i) = 1$

$$\Rightarrow \varphi_i(v_i) = v_i - \frac{1 - v_i}{1} = 2v_i - 1$$

uniform $(0, \frac{1}{2})$: $F_i(v_i) = 2v_i \quad f_i(v_i) = 2$

$$\Rightarrow \varphi_i(v_i) = v_i - \frac{1-2v_i}{2} = 2v_i - \frac{1}{2}$$

Note: unlike true valuation, could be negative!

<u>Thm</u>: Let x monotone allocation rule,
p from Myerson

Then
$$\underbrace{\mathbb{E}_{v \sim F_1 \times \dots \times F_n}\left[\sum_{i=1}^{n} p_i(v)\right]}_{\text{Expected revenue}} = \underbrace{\mathbb{E}_{v \sim F_1 \times \dots \times F_n}\left[\sum_{i=1}^{n} \varphi_i(v_i)\, x_i(v)\right]}_{\text{virtual welfare}}$$

So obvious mechanism: maximize expected virtual welfare (surplus)!

$$x(v) = \operatorname*{argmax}_{x \in X} \left(\sum_{i=1}^{n} \varphi_i(v_i)\, x_i(v) \right)$$

<u>Def</u>: A distribution F is regular if $v - \frac{1-F(v)}{f(v)}$ is nondecreasing in v

Thm: If F_i regular $\forall i \in [n]$, then virtual welfare maximizing allocation rule is monotone.

Pf sketch:

F_i's regular $\Rightarrow$ higher bids result in higher virtual values

$\Rightarrow$ same logic as for surplus maximization

Ex: Single item, n bidders each with $F_i = F$ (regular)

what is virtual welfare-maximizing allocation rule?

$$\max_{x \in \{0,1\}^n} \sum_{i=1}^{n} \varphi(v_i)\, x_i(v) \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i \leq 1$$

Give to bidder with highest virtual valuation

$\Rightarrow$ bidder with highest valuation

$\Rightarrow$ price is second-highest, by Myerson

virtual values can be negative!

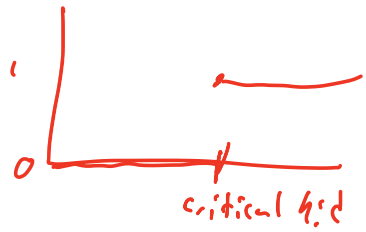
If all negative, best not to give item to anyone!

Virtual welfare maximizing allocation:

Let i^* be highest bidder.

If $\varphi_{i^*}(v_{i^*}) > 0$, give item to i^*

Else keep item



Price from Myerson:

$$\max(\text{second highest bid}, \varphi^{-1}(0))$$

Vickrey auction with reserve price $\varphi^{-1}(0)$!

If $F = \text{Uniform}(0,1)$, $\varphi^{-1}(0) = \frac{1}{2}$ $\quad \varphi(v_i) = 2v_i - 1$

$\Rightarrow$ reserve price $= 1/2$

Thm: Let x monotone allocation rule,
p from Myerson

Then
$$\mathop{E}_{v \sim F_1 \times \dots \times F_n}\left[\sum_{i=1}^{n} p_i(v)\right] = \mathop{E}_{v \sim F_1 \times \dots \times F_n}\left[\sum_{i=1}^{n} \varphi_i(v_i)\, x_i(v)\right]$$

Pf:

Myerson: $p_i(v) = \int_0^{v_i} z \cdot x_i'(v_{-i}, z)\, dz$

Fix $i \in [n]$, v_{-i}

$$\mathop{E}_{v_i \sim F_i}\left[p_i(v_i)\right] = \int_0^{v_{max}} p_i(v_{-i}, v_i)\, f_i(v_i)\, dv_i \qquad \text{(def of expectation)}$$

$$= \int_0^{v_{max}} \left(\int_0^{v_i} z \cdot x_i'(v_{-i}, z)\, dz \right) f_i(v_i)\, dv_i \qquad \text{(Myerson)}$$

$$= \int_0^{v_{max}} \left(\int_z^{v_{max}} f_i(v_i)\, dv_i \right) \cdot z \cdot x_i'(v_{-i}, z)\, dz \qquad \text{reverse order of integration}$$

$$= \int_0^{v_{max}} \underbrace{(1 - F_i(z)) \cdot z}_{f} \cdot \underbrace{x_i'(v_{-i}, z)}_{g'}\, dz$$

Integration by parts: $\int f g'\, dx = fg - \int g f'\, dx$

$$g'(z) = x_i'(v_{-i}, z) \Rightarrow g(z) = x_i(v_{-i}, z)$$

$$f(z) = (1 - F_i(z)) \cdot z$$

$$\Rightarrow f'(z) = 1 - F_i(z) - z f_i(z)$$

$$= \cancel{\Big[(1 - F_i(z)) z \cdot x_i(v_{-i}, z) \Big]_0^{v_{max}}}$$

$$- \int_0^{v_{max}} x_i(v_{-i}, z) (1 - F_i(z) - z f_i(z)) \, dz$$

$$= - \int_0^{v_{max}} x_i(v_{-i}, z) (1 - F_i(z) - z f_i(z)) \, dz$$

$$= \int_0^{v_{max}} x_i(v_{-i}, z) \left(z - \frac{1 - F_i(z)}{f_i(z)} \right) f_i(z) \, dz$$

$$= \int_0^{v_{max}} x_i(v_{-i}, z) \, \varphi_i(z) \, f_i(z) \, dz$$

$$= \mathop{E}_{z \sim F_i} \left[\varphi_i(z) \, x_i(v_{-i}, z) \right]$$

$$= \mathop{E}_{v_i \sim F_i} \left[\varphi_i(v_i) \, x_i(v) \right]$$

$$\Rightarrow \mathop{E}_{v \sim F_1 \times \dots \times F_n} \left[\sum_{i=1}^{n} p_i(v) \right] = \sum_{i=1}^{n} \mathop{E}_{v \sim F_1 \times \dots \times F_n} \left[p_i(v) \right]$$

$$= \sum_{i=1}^{n} \mathop{E}_{v \sim F_1 \times \dots \times F_n} \left[\varphi_i(v_i)\, x_i(v) \right]$$

$$= \mathop{E}_{v \sim F_1 \times \dots \times F_n} \left[\sum_{i=1}^{n} \varphi_i(v_i)\, x_i(v) \right]$$