Revenue - Maximizing Auctions

Note: start thinking about course projects!

Still want incentive compatibility, but want to maximize revenue rather than social surplus.

Trivial example: single-item, single-bidder

"posted price mechanism"

Need to make some extra assumptions!
Setup:
- Single-parameter environment: bidders \( C_n \), feasible set \( X \subseteq \mathbb{R}^{\geq 0} \)
- For each \( i \in C_n \) distribution \( F_i \):
  - support of \( F_i \) \( \subseteq [0, v_{\text{max}}] \)
  - Also use \( F_i \) to denote CDF (cumulative distribution function):
    \[
    F_i(2) = \Pr_{x \sim F_i}(x \leq 2) \quad F_i(v_{\text{max}}) = 1
    \]
  - \( F_i \) is probability density function:
    \[
    \int_0^2 F_i(x) \, dx = F_i(2)
    \]
- Private \( v_i \sim F_i \)
- Goal: IC mechanism maximizing expected revenue

Examples:

Single-item, single-bidder:

Set price \( r \Rightarrow \mathbb{E}[\text{Revenue}] = r(1 - F(r)) \)

\( F_i \sim \text{Uniform}(a_i, 1) \); set \( r = \frac{1}{2}, \mathbb{E}[\text{Revenue}] = \frac{1}{4} \)

\( F(x) = x \)
**Single-item Two bidders:**

\[ S \in \{ F_1, F_2 \text{ uniform } (0, 1) \]  

**Second-price (Vickrey) auction:**

\[ E[C_{\text{revenue}}] = E[C_{\text{second-highest valuation}}] = \frac{1}{3} \]

**"Reserve price r" auction:** Give item to highest bidder if highest bid \( \geq r \).
Otherwise, no one gets item.

**Price from Myerson:**

\[ \max (r, \text{second-highest}) \]

\[ r = \frac{1}{2} \]

\[ E[C_{\text{revenue}}] = \Pr[\text{both above } \frac{1}{2}] \cdot E[C_{\text{second-highest} | \text{both above } \frac{1}{2}}] \]

\[ + \Pr[\text{highest } \geq \frac{3}{2}, \text{second } < \frac{1}{2}] \cdot \frac{1}{2} \]

\[ = \frac{1}{4} \cdot \frac{5}{3} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{12} > \frac{1}{3} \]
Back to main setting:

Single-parameter environment \( \Rightarrow \) Myerson applies
\[ \Rightarrow \text{for allocation function } x : \{0, \ldots, \text{num}\}^n \rightarrow X, \] price \( p \) from Myerson
\[ \Rightarrow \text{Revenue maximizing auction is monotone } x, p \] from Myerson maximizing
\[ \mathbb{E} \sum_{i=1}^{n} \mathbb{E} \left[ \Pi_i (v) \right] \]

**Def:** The *virtual valuation of hidden* \( i \) with valuation \( v_i \) is:
\[ \varphi_i (v_i) = v_i - \frac{1 - F_i (v_i)}{F_i^* (v_i)} \]

**Ex:** \( F: \text{uniform}\{0, 1\} : F_i (v_i) = v_i, \; F_i^* (v_i) = 1 \)
\[ \Rightarrow \varphi_i (v_i) = v_i - \frac{1-v_i}{1} = 2v_i - 1 \]
uniform \( u_i^* \) : \( F_i(u_i) = 2u_i \) \( F_i(u_i) = 2 \)
\( \Rightarrow \) \( \varphi_i(u_i) = u_i - \frac{(u_i - u_i^*)}{2} = 2u_i - \frac{1}{2} \)

Note: unlike true valuations, could be negative!

Then: let \( x \) monotone allocation rule, \( p \) from Myerson

Then
\[
\mathbb{E} \left[ \sum_{i=1}^{n} \varphi_i(u_i) x_i(u_i) \right] = \mathbb{E} \left[ \sum_{i=1}^{n} \varphi_i(u_i) x_i(u_i) \right]
\]

Expected revenue virtual welfare

So obvious mechanism: maximize expected virtual welfare (surplus)!

\[
x(u) = \arg\max_{x \in X} \left( \sum_{i=1}^{n} \varphi_i(u_i) x_i(u) \right)
\]

Def: A distribution \( F \) is regular if \( v = \frac{1 - F(u)}{F(u)} \) is nondecreasing in \( u \)
Thm: If \( F_i \) regular \( \forall i \in [n] \), then virtual welfare maximizing allocation rule is monotone.

**Proof sketch:**

- \( F_i \)’s regular \( \Rightarrow \) higher bids result in higher virtual values
- \( \Rightarrow \) same logic as for surplus maximization

Ex: Single item, \( n \) bidders each with \( F_i = F \) (regular)

What is virtual welfare-maximizing allocation rule?

\[
\max_{x \in \{0,1\}^n} \sum_{i=1}^n q(v_i) x_i(v_i) \quad \text{s.t.} \quad \sum_{i=1}^n x_i \leq 1
\]

Give to bidder with highest virtual valuation

\( \Rightarrow \) bidder with highest valuation

\( \Rightarrow \) price is second-highest, by Myerson

Virtual values can be negative!

If all negative, best not to give item to anyone!
virtual welfare maximizing allocation:
Let \( i^* \) be the highest bidder.
If \( \varphi_{i^*}(v_{i^*}) > 0 \), give item to \( i^* \)
Else keep item.

Price from Myerson:
\[
\max \left( \text{second highest bid}, \varphi^{-1}(0) \right)
\]

Vickrey auction with reserve price \( \varphi^{-1}(0) \).

\( f = \text{Uniform} (0,1) \), \( \varphi^{-1}(0) = \frac{1}{2} \)
\( \varphi(v_i) = 2v_i - 1 \)
\( \Rightarrow \) reserve price = \( \frac{1}{2} \)
Then, Let $x$ monotonically allocation only $p$ from Myerson

$$ E \left[ \sum_{i=1}^{n} p_i(u) \right] = E \left[ \sum_{i=1}^{n} \phi_i(u) x_i(u) \right] $$

**PF:**

**Myerson:** $p_i(u) = \int_0^{u_i} z \cdot x_i(u, z) \, dz$

$F; x \, i \in \mathcal{W}, \, u_-$

$$ E \left[ p_i(u) \right] = \int_0^{u_{\mathcal{W}}} p_i(u, v_i) \, f_i(v_i) \, dv_i \quad \text{(law of expectation)} $$

$$ = \int_0^{u_{\mathcal{W}}} \left( \int_0^{u_i} z \cdot x_i(u, z) \, dz \right) \, f_i(v_i) \, dv_i \quad \text{(Myerson)} $$

$$ = \int_0^{u_{\mathcal{W}}} \left( \int_0^{u_i} f_i(v_i) \, dv_i \right) \cdot z \cdot x_i(u, z) \, dz \quad \text{reverse order of integration} $$

$$ = \int_0^{u_{\mathcal{W}}} \left( 1 - F_i(z) \right) \cdot z \cdot x_i(u, z) \, dz \quad \text{Integration by parts: } \int f g' \, dx = fg - \int g f' \, dx $$
\[ g(z) = x_i(v_{-i}, z) \rightarrow g(z) = x_i(v_{-i}, z) \]
\[ f(z) = (1 - F_i(z)) \cdot z \]
\[ \Rightarrow f'(z) = 1 - F_i(z) - z F_i(z) \]

\[
= \left[ (1 - F_i(z))z \ x_i(v_{-i}, z) \right]_{0}^{u_{max}} \\
- \int_{0}^{u_{max}} x_i(v_{-i}, z) \ (1 - F_i(z) - z F_i(z)) \ dz \\
= -\int_{0}^{u_{max}} x_i(v_{-i}, z) \ (1 - F_i(z) - z F_i(z)) \ dz \\
= \int_{0}^{u_{max}} x_i(v_{-i}, z) \ (z - \frac{1 - F_i(z)}{F_i(z)}) \ F_i(z) \ dz \\
= \int_{0}^{u_{max}} x_i(v_{-i}, z) \ q_i(z) \ F_i(z) \ dz \\
= \mathbb{E}_{\exists z \sim F_i} \left[ q_i(z) x_i(v_{-i}, z) \right] \\
= \mathbb{E}_{v_i \sim F_i} \left[ q_i(v_i) x_i(v) \right]
\[ E \left( \prod_{i=1}^{n} p_i(u) \right) = \prod_{i=1}^{n} E \left[ p_i(u) \right] \]

\[ = \prod_{i=1}^{n} \mathbb{E} \left[ q_i(u_i) x_i(u) \right] \]

\[ = \mathbb{E} \left[ \prod_{i=1}^{n} q_i(u_i) x_i(u) \right] \]