3/17/22: Single-Parameter Environments and Myerson's Lemma

Single-Parameter Environment

- Bidders $[n]$
  - Bidder i has private valuation v_i: "value per unit of stuff"
- Feasible set $X \subseteq \mathbb{R}^n$. (not necessarily "nice" in any way)
- Given $x \in X$:
  - x_i = amount of "stuff" bidder i gets
  - Social surplus of $x \in X$ is $\sum_{i=1}^{n} v_i x_i$

- Single item: $X = \{x \in \{0,1\}^n : \sum_{i=1}^{n} x_i = 1\}$

- Sponsored search: (k slots, CTR $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k$, bidder i gets $\alpha_j v_i$ from winning slot j)

$$X = \{x \in \mathbb{R}^n \text{ s.t. each entry of } x \text{ either } 0 \text{ or some } \alpha_i, \text{ each } \alpha_i \text{ appears at most once}\}$$

Sealed-bid auction:

- Collect bids $b = (b_1, b_2, \dots, b_n)$ from bidders
- Choose feasible allocation $x(b) \in X$.

  (notation: $x_i(b) = x(b)_i$) allocation rule $x : \mathbb{R}^n_{\geq 0} \to X$
- Choose payments $p(b) \in \mathbb{R}^n_{\geq 0}$

  ($p_i(b) = p(b)_i$) payment rule $p : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$
- Quasilinear utilities: $u_i(b) = v_i x_i(b) - p_i(b)$
- Nonnegative utilities: $p_i(b) \in [0, b_i x_i(b)]$

  (for truthful bidders)

so mechanism is pair (x, p): (allocation rule, payment rule)

Def: Allocation rule x is implementable if there is a payment rule p s.t. (x, p) is incentive compatible

Q: what allocation rules are implementable?

- Single-item: give to highest bidder: ✓
  - give to second highest?
- Sponsored search: greedy assignment?

Def: Allocation rule $x : \mathbb{R}^n_{\geq 0} \to X$ is monotone if for all $i \in [n]$ and other bids b_{-i}, allocation $x_i(b_{-i}, z)$ is nondecreasing in z

Thm [Myerson]:

1) Allocation rule x is implementable if and only if x monotone
2) If x monotone, there is unique payment rule p s.t. (x, p) is incentive compatible
3) unique payment rule given by explicit formula

Pf (sketch):

Implementable $\Rightarrow$ Monotone:

Let p payment rule s.t. (x, p) IC.

Fix $i \in [n]$, other bids b_{-i}.

Simplify notation: $x_i(z) = x_i(z, b_{-i})$
$p_i(z) = p_i(z, b_{-i})$

WTS: x_i nondecreasing

Let $y > z \geq 0$. WTS $x_i(y) \geq x_i(z)$

If $v_i = z$:

$$\underbrace{z \cdot x_i(z) - p_i(z)}_{\text{utility of truth}} \geq \underbrace{z \cdot x_i(y) - p_i(y)}_{\text{utility of bidding } y}$$

$$\Rightarrow p_i(y) - p_i(z) \geq z\,(x_i(y) - x_i(z))$$

If $v_i = y$:

$$\underbrace{y \cdot x_i(y) - p_i(y)}_{\text{utility of truth}} \geq \underbrace{y \cdot x_i(z) - p_i(z)}_{\text{utility of bidding } z}$$

$$\Rightarrow p_i(y) - p_i(z) \leq y\,(x_i(y) - x_i(z))$$

$$\Rightarrow z\,(x_i(y) - x_i(z)) \leq p_i(y) - p_i(z) \leq y\,(x_i(y) - x_i(z))$$

$$\Rightarrow \underbrace{(y-z)}_{>0}\,(x_i(y) - x_i(z)) \geq 0$$

$$\Rightarrow x_i(y) - x_i(z) \geq 0$$

$\Rightarrow x$ monotone

Monotone $\Rightarrow$ Implementable:

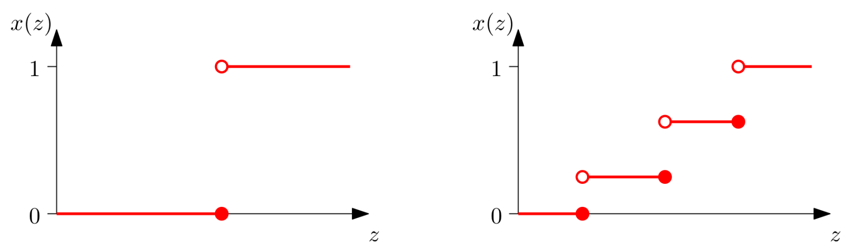
WTS $\exists\, p$ s.t. (x, p) IC

Let $i \in [n]$, other bids b_{-i}.

$x_i(z) = x_i(b_{-i}, z)$

What should $p_i(z) = p_i(b_{-i}, z)$ be?

Easy case: piecewise constant (monotone)



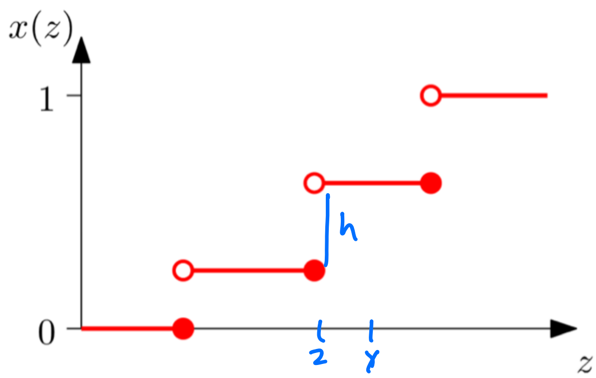
(a) 0-1 Monotone Curve

(b) Piecewise Constant Monotone Curve

(single item) (sponsored search)

Inequalities: need to have

$$z(x_i(y) - x_i(z)) \le p_i(y) - p_i(z) \le y(x_i(y) - x_i(z))$$



(b) Piecewise Constant Monotone Curve

y approaching z from above

z not breakpoint: $x_i(y) = x_i(z)$

$\Rightarrow p_i(y) = p_i(z)$

$\Rightarrow$ on "flat parts", price is the same

z a breakpoint: $\lim_{y \to z^+} (x_i(y) - x_i(z)) = h$

$\Rightarrow \lim_{y \to z^+} (p_i(y) - p_i(z)) = zh$

$\Rightarrow$ at breakpoint, price increases by

(value of breakpoint) $\times$ (increase in allocation at breakpoint)

$p_i(0) = 0$ (since nonnegative utilities, prices ≥ 0)

Sps $z_1, z_2, \ldots, z_k$ breakpoints of x_i

$\Rightarrow p_i(z) = \sum_{j=1}^{\ell} z_j \cdot (\text{jump in } x_i \text{ at } z_j)$

$\lim_{y \to z_j^+} \left(x_i(y) - x_i(z) \right)$

ℓ s.t. $z_\ell < z \leq z_{\ell+1}$

If x is implementable, must be with this p

Given bids b, for each i, compute $p_i(b_i)$

More general monotone allocation functions:

If x differentiable:

$$p_i(z) = \int_0^z \underbrace{a}_{\text{value at } a} \cdot \underbrace{\frac{d}{da} x_i(a)\, da}_{\text{jump in allocation at } a}$$

Can extend to non-differentiable with calculus tricks

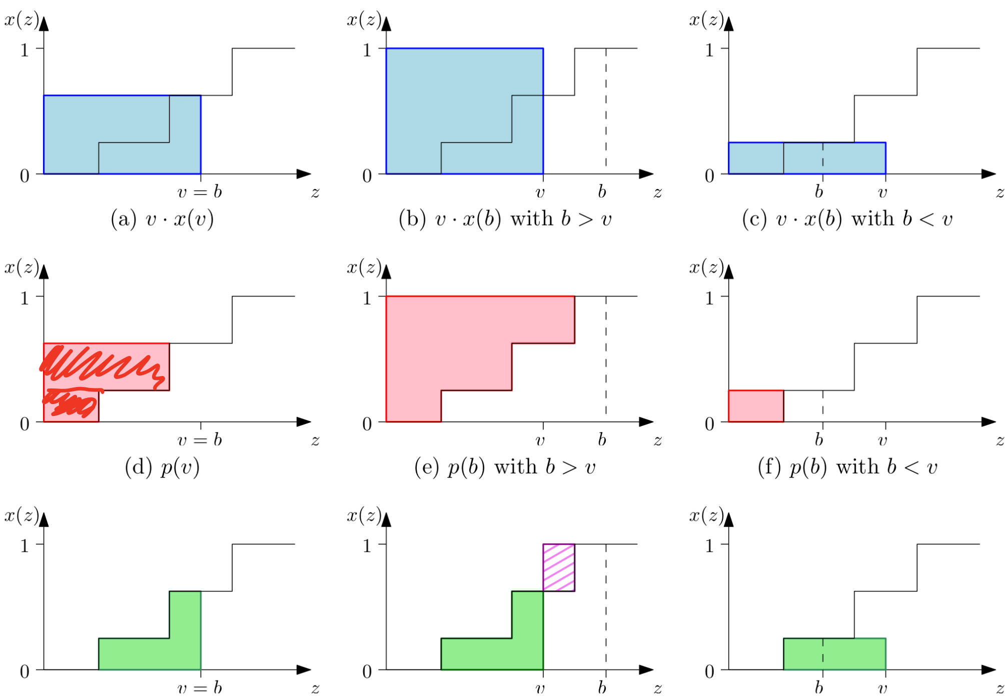
$\Rightarrow$ can compute prices as long as can compute appropriate derivatives and integrals (part 3 of Myerson)

- Unique (part 2 of Myerson)

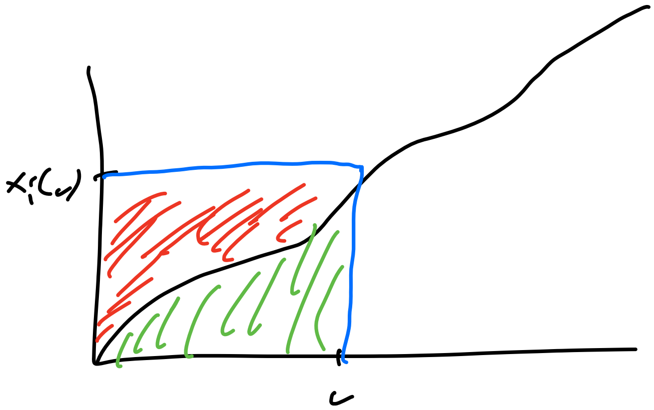
Still need to prove that (x, p) incentive compatible

Proof by picture for piecewise constant case:

Truth overbidding underbidding



(a) $v \cdot x(v)$
(b) $v \cdot x(b)$ with $b > v$
(c) $v \cdot x(b)$ with $b < v$
(d) $p(v)$
(e) $p(b)$ with $b > v$
(f) $p(b)$ with $b < v$
(g) utility with $b = v$
(h) utility with $b > v$
(i) utility with $b < v$



Myerson for Sponsored Search!

Given bids, sort, reindex so $b_1 \geq b_2 \geq \dots \geq b_n$

$$p_j(b) = \sum_{l=j}^{k} b_{l+1}(\alpha_l - \alpha_{l+1})$$

Not generalized second price! $p_j(b) = b_{j+1}$ if $j \leq k$

By uniqueness, GSP not IC

See Roughgarden problems for equilibrium-based analysis of GSP