13.1 Connection Game: Introduction

Today we're going to talk about a different game known as the *connection game*, which is one of the simplest "network formation games". Like with most of the other game we've talked about, we care about the game because it is interesting and important, but also to demonstrate a new concept or type of analysis. We'll use the connection game to talk about the price of *stability* (rather than anarchy) and to introduce a new type of equilibrium known as a *strong Nash*.

Let's start by defining the game.

- Directed graph $G = (V, E)$.
- Edge costs $c_e \in \mathbb{R}_{\geq 0}$ for each $e \in E$.
- k players, player i has a a source/sink pair $(s_i, t_i) \in V \times V$
- Strategies for player i are $\mathcal{P}_i = \{s_i \to t_i \text{ paths}\}$. Let $S = \mathcal{P}_1 \times \cdots \times \mathcal{P}_k$.
- The cost of every edge is split among the edges using it. More formally:
  - Let $s = (P_1, P_2, \dots, P_k) \in S$
  - For each $e \in E$, let $k_e(s) = |\{i : e \in P_i\}|$.
  - Then we define player costs as $C_i(s) = \sum_{e \in P_i} \frac{c_e}{k_e(s)}$.
- Our global cost will be $\text{cost}(s) = \sum_{i=1}^k C_i(s) = \sum_{e \in \cup_{i=1}^k P_i} c_e$.

It's pretty easy to see that the price of anarchy can be quite bad. Consider the following simple example: there are two nodes s and t, and all players have source s and sink t. There are two edges from 2 to t, the top with cost 1 and the bottom with cost k. Then OPT is for all players to use the top edge, for a global cost of 1. But if all players use the bottom edge then they all split its cost, so they all pay 1, and thus none of them have incentive to deviate to the top edge. Thus all players using the bottom edge is a Nash with global cost k. So the price of anarchy is at least k.

13.2 Price of Stability

Even though the price of anarchy is bad, what about the price of stability? Recall that the price of stability is the ratio of the *best* Nash to OPT, rather than the worst Nash. As a side note, this means that as we expand our notion of equilibrium (from pure Nash, to mixed, to correlated, to coarse correlated) the price of stability might get better, and it never gets worse (the opposite of the price of anarchy). So, in particular, if we can show that there's a good pure Nash, then we'll have bounded the price of stability for all notions of equilibrium which include pure Nash.

Consider the following example. There are k players, each one of which has their own source s_i. There is an edge from each s_i to a global sink t, with the cost of this edge being $1/i$. There is one other vertex v, and there is an edge of cost 0 from each s_i to v. Finally, there is an edge of cost $(1+\epsilon)$ from v to t.

OPT in this game is for every player to go from their source to v (along a zero-cost edge) and then to t, for a total cost of $1+\epsilon$. But in this solution player k is paying $(1+\epsilon)/k$, while if they switch to the edge (s_k, t) they would pay $1/k$. Once they've switched, the same logic will hold for player $k-1$, etc. So it's not too hard to see that the unique pure Nash is when every player uses their direct edge. More formally, suppose that there is some set $A \subseteq [k]$ of players that are using the path through v in some strategy profile s. Let i be the maximum value in A. Then $C_i(s) = \frac{1+\epsilon}{|A|} \geq \frac{1+\epsilon}{i}$. Hence s is not a pure Nash, since player i has incentive to switch.

So the only pure Nash is when all players use their direct edge. The total cost of this Nash is $\sum_{i=1}^{k} \frac{1}{i} = H_k = \Theta(\ln k)$. Hence the price of stability in this example is only H_k (the k'th harmonic number). The implies that in general, the price of stability in the connection game is $\Omega(\ln n)$.

It turns out that this is the *exactly* the worst case, even up to constants!

Theorem 13.2.1. *The price of stability in any connection game is at most H_k.*

Before we prove this theorem, it will be useful for us to prove that this is actually a potential game. This is useful not only because it implies that there is a pure Nash, but because the potential function itself will be useful when we prove Theorem 13.2.1. We'll prove that the following function is a potential function (where recall that $H_0 = 0$):

$$\Psi_e(s) = c_e \cdot H_{k_e(s)} \qquad \forall e \in E$$
$$\Psi(s) = \sum_{e \in E} \Psi_e(s)$$

Recall that in order to be a potential function, we need to prove that the change in the potential function when (exactly) one player deviates is equal to the change in cost of the player who deviates. So let's prove this.

Lemma 13.2.2. *Let $s = (P_1, P_2, \ldots, P_k) \in S$, and let $P_i' \in \mathcal{P}_i$. Then*

$$\Psi(s) - \Psi(s_{-i}, P_i') = C_i(s) - C_i(s_{-i}, P_i')$$

Proof.

$$\begin{aligned}
\Psi(s) - \Psi(s_{-i}, P_i') &= \sum_{e \in E} \left(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\right) \\
&= \sum_{e \in P_i \setminus P_i'} \left(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\right) + \sum_{e \in P_i' \setminus P_i} \left(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\right) \\
&\quad + \sum_{e \in P_i \cap P_i'} \left(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\right) + \sum_{e \in E \setminus (P_i \cup P_i')} \left(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\right) \\
&= \sum_{e \in P_i \setminus P_i'} \left(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\right) + \sum_{e \in P_i' \setminus P_i} \left(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\right) \\
&= \sum_{e \in P_i \setminus P_i'} \left(c_e H_{k_e(s)} - c_e H_{k_e(s)-1}\right) + \sum_{e \in P_i' \setminus P_i} \left(c_e H_{k_e(s)} - c_e H_{k_e(s)+1}\right) \\
&= \sum_{e \in P_i \setminus P_i'} \frac{c_e}{k_e(s)} - \sum_{e \in P_i' \setminus P_i} \frac{c_e}{k_e(s)+1} \\
&= \sum_{e \in P_i \setminus P_i'} \frac{c_e}{k_e(s)} - \sum_{e \in P_i' \setminus P_i} \frac{c_e}{k_e(s_{-i}, P_i')} \\
&= \sum_{e \in P_i} \frac{c_e}{k_e(s)} - \sum_{e \in P_i'} \frac{c_e}{k_e(s_{-i}, P_i')} \\
&= C_i(s) - C_i(s_{-i}, P_i')
\end{aligned}$$

□

Thus the connection game is a potential game, and so in particular any strategy profile that minimizes Ψ must be a pure Nash equilibrium (if any player had incentive to deviate then that deviation would result in a strategy profile which makes Ψ smaller).

One important thing to note about this lemma: it holds even if we replace P_i' by $\varnothing$. In other words, the change in the potential is equal to $C_i(s) - 0$ if we "pretend" to allow player i to deviate to not play a path at all (which of course isn't actually allowed in the game).

Let's relate this potential function to our global cost function.

Lemma 13.2.3. *$cost(s) \le \Psi(s) \le H_k \cdot cost(s)$ for all $s \in S$.*

Proof.

$$\text{cost}(s) = \sum_{e \in \cup_{i=1}^k P_i} c_e \le \sum_{e \in E} c_e H_{k_e(s)} = \Psi(s) = \sum_{e \in \cup_{i=1}^k P_i} c_e H_{k_e(s)} \le H_k \sum_{e \in \cup_{i=1}^k P_i} c_e = H_k \cdot \text{cost}(s)$$

□

Now we can finally prove Theorem 13.2.1. Let s be a global minimizer of Ψ (and thus a pure Nash). Let s^* be the optimum (the global minimizer of cost). Then

$$\text{cost}(s) \le \Psi(s) \le \Psi(s^*) \le H_k \cdot \text{cost}(s^*)$$

13.3 Strong Nash

What can we do if still want to analyze worst-case equilibria? How can we get around the bad example for the price of anarchy? One way would be to even further restrict our notion of equilibrium to a subset of the pure Nash, and hope that this subset doesn't include any bad equilibria. Of course, we don't want an arbitrary subset, but one which makes some kind of game-theoretic sense.

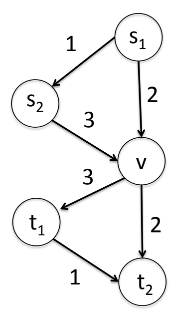
To get some intuition, let's think about what kind of extra criterion would get rid of the bad example we started with. Intuitively, the bad Nash is extremely "unstable": while no one wants to individually deviate to the top edge, if *two* players deviated to the top edge then they would each pay 1/2, and so would be happier. In other words, if we allow "coalitions" of size two, then this bad Nash equilibrium is actually not an equilibrium (while the very good Nash equilibrium where everyone uses the bottom edge is still an equilibrium). Let's try to formalize this intuition.

Definition 13.3.1. *Let $s \in S$. The strategies $s'_A \in \prod_{i \in A} S_i$ are a* beneficial deviation *for $A \subseteq [k]$ if $C_i(s_{-A}, s'_A) \le C_i(s)$ for all $i \in A$, with strict inequality for at least one $i \in A$.*

Definition 13.3.2. *$s \in S$ is a* strong Nash *if there are no beneficial deviations for ay $A \subseteq [k]$.*

Note that any strong Nash is also a pure Nash, since a pure Nash can be though of as the special case where we only care about beneficial deviations for sets (coalitions) of size 1.

Unfortunately, strong Nash equilibria do not always exist, even in the connection game. Consider the following example.



Each player has two possible strategies: a 2-hop path and a 3-hop path. Let's consider the different strategy profiles and show that none of them are a strong Nash.

- If both players use their 2-hop paths then their paths d no overlap, so they each have a cost of five. But if they both deviated to their three-hop paths then they would be overlapping on (s_2, v) and (v, t_1), so they each would have cost four.
- If they both use their 3-hop paths then they would both pay 4. But then if player 1 deviated to their 2-hop path they would instead of $2 + (3/2) = 3.5$. So player 1 has incentive to deviate.

- If one of them is using a 2-hop path and the other is using a 3-hop path, then whoever is using the 2-hop path has cost 3.5 and whoever is using the 3-hop path has cost 5.5. So the player using a 3-hop path has incentive to deviate to the 2-hop path (to have cost 4).

Thus there is no strong Nash, even though both using their 2-hop paths is a pure Nash.

This is unfortunate. However, we might get lucky and there might exist a strong Nash. And there are some interesting cases where we can guarantee that there is a pure Nash.

Theorem 13.3.3. *If $s_i = s$ and $t_i =$ for all $i \in [k]$ (all players have same source and destination) there there is a strong Nash equilibrium.*

Proof. Let s be a strategy profile in which every player chooses the same shortest $s \to t$ path. If any group of players deviate, then they are on a longer path with fewer players to split the cost of the edges, and so will be paying more. Thus there are no beneficial deviations for any coalition. □

Now let's show that *if* there is a strong Nash, then it must be relatively close to optimal in the connection game.

Theorem 13.3.4. *In the connection game, every strong Nash s has cost$(s) \leq H_k \cdot$ cost(s^*) for all $s^* \in S$ (the* strong price of anarchy *is at most H_k).*

Proof. Let $s = (P_1, P_2, \dots, P_k) \in S$ be a strong Nash and let $s^* = (P_1^*, P_2^*, \dots, P_k^*) \in S$. Since s is a strong Nash, there are no beneficial deviations for any coalition, and in particular no coalition wants to deviate to the strategies that they play in s^*. So consider the coalition $A_k = [k]$. Then there is some $i_k \in A_k$ such that $C_{i_k}(s) \leq C_{i_k}(s^*)$. Let's remove i_k from the coalition to get $A_{k-1} = A_k \setminus \{i_k\}$.

But now A_{k-1} is a coalition which does not want to deviate to s^*, so there is some $i_{k-1} \in A_k$ so that $C_{i_{k-1}}(s) \leq C_{i_{k-1}}(s_{-A_{k-1}}, s^*_{A_{k-1}})$. Let's remove i_{k-1} from A_{k-1} to get $A_{k-2} = A_{k-1} \setminus \{i_{k-1}\}$.

We can keep iterating this argument to get an ordering of the players $i_k, i_{k-1}, i_{k-2}, \dots, i_2, i_1$ so that

$$C_{i_j}(s) \leq C_{i_j}(s_{-A_j}, s^*_{A_j})$$

for all $j \in [k]$. Then we can sum up all of these inequalities to get that

$$\text{cost}(s) = \sum_{i=1}^{k} C_i(s) = \sum_{j=1}^{k} C_{i_j}(s) \leq \sum_{j=1}^{k} C_{i_j}(s_{-A_j}, s^*_{A_j})$$

Now note that $C_{i_j}(s_{-A_j}, s^*_{A_j}) \leq C_{i_j}(s^*_{A_j})$ for all j, since removing all of the players not in A_j (rather than having them play their strategies from s) means that the costs to all other players can only go up, since there are fewer players to possibly split the cost of the edges they want to buy. Thus we can extend the above inequalities to get

$$\text{cost}(s) \leq \sum_{j=1}^{k} C_{i_j}(s^*_{A_j})$$

Now let's use our potential function, but consider the "deviation" to $\varnothing$ that we discussed earlier. We get

$$\begin{aligned}
\mathrm{cost}(s) &\leq \sum_{j=1}^{k} C_{i_j}(s^*_{A_j}) = \sum_{j=1}^{k} \Big(C_{i_j}(s^*_{A_j}) - C_{i_j}(s^*_{A_j - i_j}, \varnothing) \Big) \\
&= \sum_{j=1}^{k} \Big(\Psi(s^*_{A_j}) - \Psi(s^*_{A_{j-1}}) \Big) \\
&= \Psi(s^*) - \Psi(s^*_{\varnothing}) = \Psi(s^*) \\
&\leq H_k \cdot \mathrm{cost}(s^*)
\end{aligned}$$

□