3/3/22:

Def: A cost-minimization game with objective function $cost: S \to \mathbb{R}$ is (λ, μ)-smooth if

1) $cost(s) \leq \sum_{i=1}^{k} C_i(s) \quad \forall s \in S$

2) $\sum_{i=1}^{k} C_i(s_{-i}, s_i') \leq \lambda \cdot cost(s') + \mu \cdot cost(s) \quad \forall s, s' \in S$

Switch to utility maximizing:

k players, player i has strategy set S_i, $S = S_1 \times S_2 \times \dots \times S_k$, player i has utility function $u_i: S \to \mathbb{R}$

Def: A utility-maximization game with objective function $V: S \to \mathbb{R}$ is (λ, μ)-smooth if

value

1) $V(s) \geq \sum_{i=1}^{k} u_i(s) \quad \forall s \in S$

2) $\sum_{i=1}^{k} u_i(s_{-i}, s_i') \geq \lambda \cdot V(s') - \mu \cdot V(s) \quad \forall s, s' \in S$

Price of Total Anarchy: $\dfrac{\max_{s \in S} V(s)}{\min_{\sigma \in CCE} \mathbb{E}_{s \sim \sigma}[V(s)]} = \dfrac{OPT}{\text{worst CCE}}$

Thm: PoTA in a (λ, μ)-smooth utility-maximization game is at most $\dfrac{1+\mu}{\lambda}$

Pf: Same as cost-minimization

$$\mathbb{E}_{s \sim \sigma}[V(s)] \geq \mathbb{E}_{s \sim \sigma}\left[\sum_{i=1}^{k} u_i(s)\right] = \sum_{i=1}^{k} \mathbb{E}[u_i(s)]$$

def of CCE $\rightarrow$ $$\geq \sum_{i=1}^{k} \mathbb{E}_{s \sim \sigma}[u_i(s_{-i}, s_i^*)] = \mathbb{E}_{s \sim \sigma}\left[\sum_{i=1}^{k} u_i(s_{-i}, s_i^*)\right]$$

def of smooth $\rightarrow$ $$\geq \mathbb{E}_{s \sim \sigma}[\lambda \cdot V(s^*) - \mu \cdot V(s)] = \lambda \cdot V(s^*) - \mu \cdot \mathbb{E}_{s \sim \sigma}[V(s)]$$

$$\Rightarrow V(s^*) \leq \frac{(1+\mu)}{\lambda} \mathbb{E}_{s \sim \sigma}[V(s)]$$

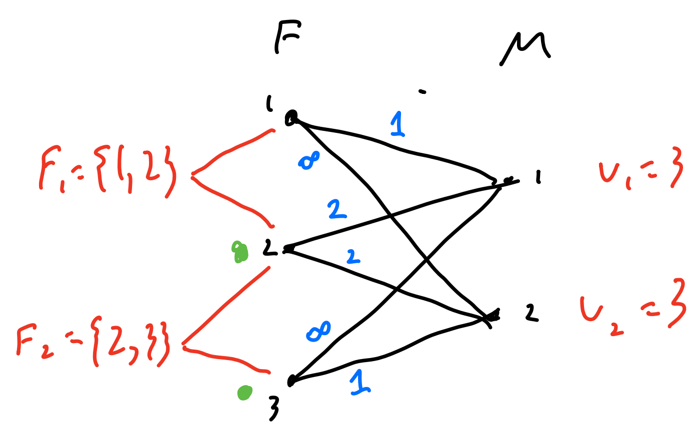
Facility Location Game:

"Competitive facility location with price-taking markets and profit-maximizing firms"

- Set F of possible locations
- k players. Player i has strategy set $F_i \subseteq F$
  (places where player i can build a facility)
- Set M of markets. Each $j \in M$ has value $v_j \geq 0$
  (amount customer at market j willing to pay for service)
- For each $x \in F$ and $j \in M$, cost $c_{x,j} \geq 0$
  (cost of serving market j from location x)

Before define utilities formally, do example.

$k=2$, $|F|=3$, $|M|=2$



Intuition: - player at location will charge each market as much as possible

- each market will accept lowest price if below v_j, give money to player
- Player who receives money, pays cost of service

$s_1 = 1$, $s_2 = 3$:

player 1: offers price ∞ to market 2
3 to market 1

player 2: offers price ∞ to market 1
3 to market 2

$\Rightarrow$ player 1 profit (utility) $3 - 1 = 2$
2 profit (utility) $3 - 1 = 2$

$s_1 = 2$, $s_2 = 3$:

If P2 still charges 3 to M2, then P1 can charge < 3 and still make a profit!

True if P2 charges > 2 to M2
$\Rightarrow$ P2 can only offer price 2 to M2
P1 still offers price 3 to M1

$\Rightarrow$ P1 utility $= 3-2=1$

P2 utility $= 2-1=1$

Formal definition of utilities:

WLOG, $c_{x,j} \leq v_j \quad \forall x \in F, j \in M$

(change example from cost ∞ to cost 3)

Strategy profile s

$\Rightarrow$ player i at location s_i

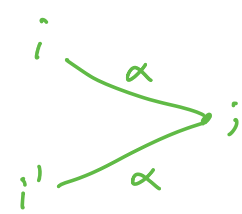
who will end up servicing market j?

- whoever is at closest location!

$$c_{s_i,j} \leq c_{s_x,j} \quad \forall x \in [k]$$

what price do they charge?

- Second-closest!



$$P_{i,j}(s) = \min_{x \neq i} c_{s_x,j}$$

Profit of player i from market j:

$$\pi_{i,j}(s) = \begin{cases} p_{i,j}(s) - c_{s_i,j} & \text{if } c_{s_i,j} \leq c_{s_x,j} \ \forall x \in [k] \\ 0 & \text{otherwise} \end{cases}$$

utility of player i under strategy profile s:

$$u_i(s) = \sum_{j \in M} \pi_{i,j}(s)$$

<u>Global Value Function</u> $V : S \to \mathbb{R}$

sum total happiness over all players and markets.

"social surplus", "social welfare"

Under strategy profile s, let

$f(j)$ = player who serves market j

$\Rightarrow p_{f(j),j}(s)$ = price paid by market j

Total player utilities:

$$\sum_{i=1}^{k} u_i(s) = \sum_{i=1}^{k} \sum_{j \in M} \pi_{i,j}(s) = \sum_{i=1}^{k} \sum_{j : f(j)=i} \pi_{i,j}(s)$$

$$= \sum_{j \in M} \pi_{f(j),j}(s) = \sum_{j \in M} \left(p_{f(j),j}(s) - c_{s_{f(j)},j} \right)$$

"utility" of market: value - price

$\Rightarrow$ total utility of all markets =

$$\sum_{j \in M} \left(v_j - p_{r(j),j}(s) \right)$$

Add players + markets:

$$V(s) = \sum_{j \in M} \left(p_{r(j),j}(s) - c_{s_{r(j)},j} \right) + \sum_{j \in M} \left(v_j - p_{r(j),j}(s) \right)$$

$$= \sum_{j \in M} \left(v_j - c_{s_{r(j)},j} \right)$$

Prices don't matter!

<u>Smoothness</u>:

<u>Fact</u>: Potential game (exercise in Roughgarden)

<u>Thm</u>: (1,1)-smooth

$\Rightarrow$ PoTA ≤ 2

<u>Property 1</u>: $\sum_{i=1}^{k} u_i(s) \leq V(s) \quad \forall s \in S.$

Easy: added market utilities, nonnegative

<u>Property 2</u>: $u_i(s) = V(s) - V(s_{-i}) \quad \forall i \in [k], \forall s \in S$

abusing notation: value of game without player i, on strategy profile with i'th coordinate removed

English: utility = surplus created

For each $j \in M$:

- $g(j)$ = closest location with a player in s
- $g_{-i}(j)$ = closest location with a player in s_{-i}

$$\Rightarrow V(s) - V(s_{-i}) = \sum_{j \in M} \left(v_j - c_{g(j),j}\right) - \sum_{j \in M} \left(v_j - c_{g_{-i}(j),j}\right)$$

$$= \sum_{j \in M} \underbrace{\left(c_{g_{-i}(j),j} - c_{g(j),j}\right)}_{= \begin{cases} 0 & \text{if } s_i \neq g(j) \\ (\text{second closest}) - \text{closest} & \text{if } s_i = g(j) \end{cases}}$$

$= \pi_{i,j}(s)$

$$= \sum_{j \in M} \pi_{i,j}(s) \qquad = u_i(s)$$

<u>Property 3</u>: V is monotone and submodular

Abuse notation even more: also think of s as set of locations, so $s \subseteq F$

$\Rightarrow$ OK since V only depends on locations chosen, not players

Monotone: If $s \subseteq s'$, then $V(s) \leq V(s')$

Def of V: adding more chosen locations change v_j's, can only decrease $c_{f(j),j}$'s

submodular: If $s \subseteq s'$, then

$$V(s' \cup \{x\}) - V(s') \leq V(s \cup \{x\}) - V(s) \quad \forall x \in F$$

(decreasing marginal benefits)

By property 2, equivalent to proving

$$u_x(s' \cup \{x\}) \leq u_x(s \cup \{x\})$$ (u_x utility function of player choosing location x)

By def of u_x, equivalent to proving

$$\sum_{j \in M} \pi_{x,j}(s' \cup \{x\}) \leq \sum_{j \in M} \pi_{x,j}(s \cup \{x\})$$

Let's prove $\pi_{x,j}(s' \cup \{x\}) \leq \pi_{x,j}(s \cup \{x\}) \quad \forall j \in M$

If $\pi_{x,j}(s' \cup \{x\}) = 0$: ✓

if $\pi_{x,j}(s' \cup \{x\}) > 0$:

$\Rightarrow$ x closest open location to j in $s' \cup \{x\}$

$\Rightarrow$ x closest open location to j in $s \cup \{x\}$

Since $s \subseteq s'$, $\underbrace{p_{x,j}(s' \cup \{x\})}_{\text{second-closest in } s' \cup \{x\}} \leq \underbrace{p_{x,j}(s \cup \{x\})}_{\text{second-closest in } s \cup \{x\}}$

$$\Rightarrow \pi_{x,j}(s' \cup \{x\}) = p_{x,j}(s' \cup \{x\}) - c_{x,j}$$

$$\leq p_{x,j}(s \cup \{x\}) - c_{x,j} = \pi_{x,j}(s \cup \{x\})$$

Finally proving $(1,1)$-smooth:

$$\sum_{i=1}^{k} u_i(s_{-i}, s_i') = \sum_{i=1}^{k} \left(V(s_{-i}, s_i') - V(s_{-i}) \right) \quad \text{(property 2)}$$

$$\geq \sum_{i=1}^{k} \left(V(s \cup \{s_1', s_2', \dots, s_i'\}) - V(s \cup \{s_1', s_2', \dots, s_{i-1}'\}) \right) \quad \text{(submodularity)}$$

$$= V(s \cup s') - V(s) \quad \text{(telescoping)}$$

$$\geq V(s') - V(s) \quad \text{(monotonicity)}$$

Monotone Utility Games:

Generalization of facility location game due to Vetta '02

- Player i has strategy set S_i, $S = S_1 \times S_2 \times \dots \times S_k$,
  $u_i : S \to \mathbb{R}$.
- $A = S_1 \cup S_2 \cup \dots \cup S_k$ (locations)
- $V : 2^A \to \mathbb{R}$ (function on subsets of A)

  (like facility location, depends only on which locations chosen)

- Abuse notation: for $s \in S$, $V(s) = V\left(\bigcup_{i=1}^{k} \{s_i\}\right)$

Four properties of a monotone utility game:

1) $V(s) \geq \sum_{i=1}^{k} u_i(s) \quad \forall s \in S$

2) V submodular: $V(T' \cup \{x\}) - V(T') \leq V(T \cup \{x\}) - V(T)$

$\forall\, T \subseteq T' \subseteq A, \quad x \in A$

3) V monotone: $V(T) \leq V(T') \quad \forall\, T \subseteq T' \subseteq A$

4) utility of player at least surplus created:

$u_i(s) \geq V(s) - V(s_{-i}) \quad \forall i \in [k],\ s \in S$

(if =, a basic monotone utility game)

So we proved that facility location game a basic monotone utility game

<u>Thm</u>: Every monotone utility game is (1,1)-smooth

<u>Pf</u>: Property 1 of monotone utility $\Rightarrow$ property 1 of smooth

$$\sum_{i=1}^{k} u_i(s_{-i}, s_i') \geq \sum_{i=1}^{k} \left(V(s_{-i}, s_i') - V(s_{-i}) \right) \qquad \text{(property 4)}$$

$$\geq \sum_{i=1}^{k} \left(V(s \cup \{s_1', s_2', \ldots, s_i'\}) - V(s \cup \{s_1', s_2', \ldots, s_{i-1}'\}) \right) \qquad \text{(submodularity)}$$

$$= V(s \cup s') - V(s) \qquad \text{(telescoping)}$$

$$\geq V(s') - V(s) \qquad \text{(monotonicity)}$$

Note: Monotone utility games defined before smoothness.

Original bounds only for pure price of anarchy!