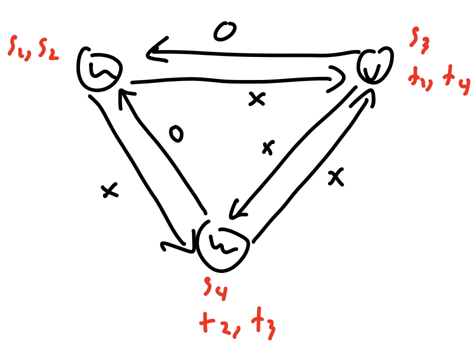
Atomic Routing Games:

- Directed graph $G=(V,E)$
- edge cost functions $c_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$
    - (continuous, nondecreasing, nonnegative)
- k players, player i has source $s_i \in V$, sink $t_i \in V$
- $P_i = \{ s_i \to t_i \text{ paths} \}$. Strategy set for player i
- $S = P_1 \times P_2 \times \dots \times P_k$
- flow is $f = (P_1, P_2, \dots, P_k) \in S$
    - $f_e = | \{ i \in [k] : e \in P_i \} |$, $c_P(f) = \sum_{e \in P} c_e(f_e)$
- Cost to player i of f is $c_{P_i}(f)$
- Global cost function: social cost $C(f) = \sum_{i=1}^{k} c_{P_i}(f) = \sum_{e \in E} f_e c_e(f_e)$

Already proved that pure Nash equilibria exist, potential game

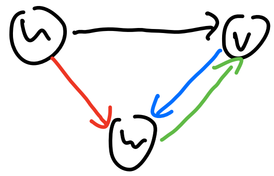
AAE example (Awerbuch, Azar, Epstein)



Each player has 2 strategies:
1-hop, 2-hop

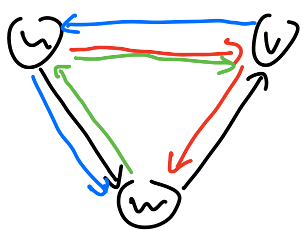
OPT: 4

Everyone uses 1-hop path:



c_1: 1
c_2: 1
c_3: 1
c_4: 1

Everyone uses 2-hop path:



c_1: 3
c_2: 3
c_3: 2
c_4: 2

If deviate:
c_1: 3
c_2: 3
c_3: 2
c_4: 2

Cost: 10

$$\Rightarrow \text{PoA} = \frac{10}{4} = \frac{5}{2} > \frac{4}{3}$$

<u>Thm</u>: In every atomic routing game with affine cost functions, Pure Price of Anarchy $\leq \frac{5}{2}$

Pf: Let $c_e(x) = a_e x + b_e$ for each $e \in E$

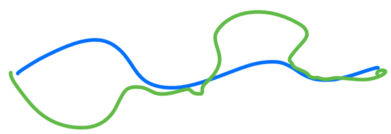
$f = (P_1, P_2, \ldots, P_k)$ pure Nash

$f^* = (P_1^*, P_2^*, \ldots, P_k^*)$ optimal flow

Since f Nash, player i does not want to switch to P_i^*:

$$\Rightarrow \sum_{e \in P_i} c_e(f_e) = \sum_{e \in P_i \cap P_i^*} c_e(f_e) + \sum_{e \in P_i^* \setminus P_i} c_e(f_e+1)$$

$$\leq \sum_{e \in P_i^*} c_e(f_e+1) \qquad (c_e \text{ nondecreasing})$$



Sum over all players i:

$$C(f) = \sum_{i=1}^{k} \sum_{e \in P_i} c_e(f_e) \leq \sum_{i=1}^{k} \sum_{e \in P_i^*} c_e(f_e+1)$$

$$= \sum_{e \in E} f_e^* \, c_e(f_e+1)$$

$$= \sum_{e \in E} f_e^* \left(a_e(f_e+1) + b_e\right) \qquad (\text{def of } c_e)$$

$$= \sum_{e \in E} \left(a_e f_e^* (f_e+1) + f_e^* b_e \right)$$

want to relate $C(f)$ to $C(f^*)$.

$\Rightarrow$ want to "disentangle" $a_e f_e^* (f_e+1)$

Fact: $\forall y, z \in \{0,1,2,3,\ldots\}$, $\quad y(z+1) \leq \frac{5}{3} y^2 + \frac{1}{3} z^2$

$$\leq \sum_{e\in E} \left(a_e\left(\tfrac{5}{3} (f_e^*)^2 + \tfrac{1}{3} f_e^2 \right) + f_e^* b_e \right)$$

$$= \sum_{e\in E} f_e^* \left(\tfrac{5}{3} a_e f_e^* + b_e \right) + \sum_{e\in E} \tfrac{1}{3} a_e f_e^2$$

$$\leq \sum_{e\in E} f_e^* \left(\tfrac{5}{3} a_e f_e^* + \tfrac{5}{3} b_e \right) + \sum_{e\in E} \tfrac{1}{3} f_e (a_e f_e + b_e)$$

$$= \sum_{e\in E} \tfrac{5}{3} f_e^* c_e(f_e^*) + \sum_{e\in E} \tfrac{1}{3} f_e c_e(f_e)$$

$$= \tfrac{5}{3} C(f^*) + \tfrac{1}{3} C(f)$$

$$\Rightarrow C(f) \leq \tfrac{5}{3} C(f^*) + \tfrac{1}{3} C(f)$$

$$\Rightarrow \tfrac{2}{3} C(f) \leq \tfrac{5}{3} C(f^*)$$

$$\Rightarrow C(f) \leq \tfrac{5}{2} C(f^*)$$

<u>Extensions</u>:

- Different players control different amounts of flow:

$$PoA \leq \frac{3+\sqrt{5}}{2} \approx 2.618$$

- degree-p polynomial costs: PoA grows exponentially with p

Smooth Games:

Observation: previous argument "canonical"

- many (not all) PoA bounds proved in similar way

Outline of previous argument:

- $s \in S$ pure Nash, s^* opt
- Objective social cost: $C(s) = \sum_{i=1}^{k} C_i(s)$

  player i's cost

- Since s Nash, $C_i(s) \leq C_i(s_{-i}, s_i^*)$

$$\Rightarrow C(s) \leq \sum_{i=1}^{k} C_i(s_{-i}, s_i^*)$$

weird "entangled" term

- Atomic Routing: $\sum_{i=1}^{k} C_i(s_{-i}, s_i^*) \leq \frac{5}{3} \sum_{i=1}^{k} C_i(s^*) + \frac{1}{3} \sum_{i=1}^{k} C_i(s)$

Just algebra!

Didn't use fact that s Nash, s^* optimal

$$\Rightarrow C(s) \leq \frac{5}{3} C(s^*) + \frac{1}{3} C(s)$$

$$\Rightarrow C(s) \leq \frac{5}{2} C(s^*)$$

Amazing thing we'll prove: If you prove a pure price of anarchy bound using this outline, then the bound you prove is also a bound on the Price of Total Anarchy

$\left(\frac{\text{worst CCE}}{\text{OPT}} \right)$

Setup: same as before, but switch to objective function

$\text{cost} : S \to \mathbb{R}$ (go beyond just social cost)

$C_i : S \to \mathbb{R} \quad \forall i \in [k]$

Def: A cost-minimization game with objective function $\text{cost}: S \to \mathbb{R}$ is (λ, μ)-smooth if

1) $\text{cost}(s) \leq \sum_{i=1}^{k} C_i(s) \quad \forall s \in S$

2) $\sum_{i=1}^{k} C_i(s_{-i}, s_i') \leq \lambda \cdot \text{cost}(s') + \mu \cdot \text{cost}(s) \quad \forall s, s' \in S$

Thm: The pure price of anarchy of a (λ, μ)-smooth game is $\leq \frac{\lambda}{1-\mu}$

Pf: Let s pure Nash

$s^* = \underset{s' \in S}{\text{argmin}}\ \text{cost}(s')$ optimal profile

$$\Rightarrow \quad cost(s) \le \sum_{i=1}^{k} C_i(s) \qquad \text{(assumption on cost)}$$

$$\le \sum_{i=1}^{k} C_i(s_{-i}, s_i^*) \qquad \text{(s a pure Nash)}$$

$$\le \lambda \cdot cost(s^*) + \mu \cdot cost(s) \qquad \text{(def of smooth)}$$

$$\Rightarrow \quad cost(s) \le \frac{\lambda}{1-\mu} \, cost(s^*)$$

<u>Thm</u>: Atomic routing games with affine cost functions are $(\frac{5}{3}, \frac{1}{3})$-smooth

<u>Pf</u>: Let $f = (P_1, P_2, \dots, P_k)$ a flow (strategy profile)

$f' = (P_1', P_2', \dots, P_k')$ different flow.

$$\sum_{i=1}^{k} C_i(f_{-i}, P_i') = \sum_{i=1}^{k} \left(\sum_{e \in P_i' \setminus P_i} c_e(f_e + 1) + \sum_{e \in P_i' \cap P_i} c_e(f_e) \right) \qquad \text{(def)}$$

$$\le \sum_{i=1}^{k} \sum_{e \in P_i'} c_e(f_e + 1) \qquad \text{(ce nondecreasing)}$$

$$= \sum_{e \in E} \sum_{i : e \in P_i'} c_e(f_e + 1) \qquad \text{(switch order of summation)}$$

$$= \sum_{e \in E} f_e' \, c_e(f_e + 1) \qquad \text{(def of } f_e' \text{)}$$

$$= \sum_{e\in E} \left(a_e f'_e (f_e + 1) + b_e f'_e \right)$$ (def of c_e)

$$\leq \sum_{e\in E} \left(a_e \left(\frac{5}{3} (f'_e)^2 + \frac{1}{3} f_e^2 \right) + b_e f'_e \right)$$ (algebra)

$$\leq \frac{5}{3} \sum_{e\in E} f'_e (a_e f'_e + f_e) + \frac{1}{3} \sum_{e\in E} f_e (a_e f_e + b_e)$$

$$= \frac{5}{3} \sum_{e\in E} f'_e \, c_e(f'_e) + \frac{1}{3} \sum_{e\in E} f_e \, c_e(f_e)$$

$$= \frac{5}{3} C(f') + \frac{1}{3} C(f)$$

Price of Total Anarchy:

Thm: The Price of Total Anarchy of a (λ,μ)-smooth game is at most $\frac{\lambda}{1-\mu}$.

Def: Distribution σ over strategy profiles S is a coarse correlated equilibrium if

$$\mathop{E}_{s\sim\sigma}\left[C_i(s)\right] \leq \mathop{E}_{s\sim\sigma}\left[C_i(s_{-i}, s_i')\right] \quad \forall i \in [k],\ \forall s_i' \in S_i$$

So WTS: If σ a CCE of (λ,μ)-smooth game and s^* optimal, then $\dfrac{\mathop{E}_{s\sim\sigma}[\text{cost}(s)]}{\text{cost}(s^*)} \leq \dfrac{\lambda}{1-\mu}$

$$\mathop{E}_{s\sim\sigma}\left[\text{cost}(s)\right] \leq \mathop{E}_{s\sim\sigma}\left[\sum_{i=1}^{k} C_i(s)\right] \qquad \text{(property 1 of smooth)}$$

$$= \sum_{i=1}^{k} \mathop{E}_{s\sim\sigma}\left[C_i(s)\right] \qquad \text{(linearity of expectation)}$$

$$\leq \sum_{i=1}^{k} \mathop{E}_{s\sim\sigma}\left[C_i(s_{-i}, s_i^*)\right] \qquad \text{(def of CCE)}$$

$$= \mathop{E}_{s\sim\sigma}\left[\sum_{i=1}^{k} C_i(s_{-i}, s_i^*)\right] \qquad \text{(linearity of expectations)}$$

$$\leq \mathop{E}_{s\sim\sigma}\left[\lambda\cdot cost(s^*) + \mu\cdot cost(s)\right]$$ (def of smooth)

$$= \lambda\cdot cost(s^*) + \mu\cdot \mathop{E}_{s\sim\sigma}[cost(s)]$$ (linearity of expectation)

$$\Rightarrow \mathop{E}_{s\sim\sigma}[cost(s)] \leq \frac{\lambda}{1-\mu}\, cost(s^*)$$

So if game is smooth (like atomic routing), all CCEs are pretty good

$\Rightarrow$ no-regret algorithms lead to pretty good behavior!

Q: But no-regret only gets us to ε-CCE, not full CCE
what about ε-CCE?

Def: A distribution σ over S is an ε-CCE if

$$\mathop{E}_{s\sim\sigma}[C_i(s)] \leq (1+\varepsilon)\mathop{E}_{s\sim\sigma}[C_i(s_{-i}, s_i')] \qquad \forall i\in[k], \forall s_i'\in S_i$$

(Different than additive definition we used before, but essentially equivalent)

Thm: For any (λ,μ)-smooth game and $\varepsilon < \frac{1}{\mu} - 1$, for every ε-CCE σ,

$$\underset{s\sim\sigma}{E}[\mathrm{cost}(s)] \le \frac{(1+\varepsilon)\lambda}{1-(1+\varepsilon)\mu}\,\mathrm{cost}(s^*)$$

Pf: Same as before

$$\underset{s\sim\sigma}{E}[\mathrm{cost}(s)] \le \underset{s\sim\sigma}{E}\left[\sum_{i=1}^{k} C_i(s)\right] = \sum_{i=1}^{k} \underset{s\sim\sigma}{E}[C_i(s)]$$

$$\le (1+\varepsilon)\sum_{i=1}^{k} \underset{s\sim\sigma}{E}[C_i(s_{-i}, s_i^*)] = (1+\varepsilon)\underset{s\sim\sigma}{E}\left[\sum_{i=1}^{k} C_i(s_{-i}, s_i^*)\right]$$

$$\le (1+\varepsilon)\underset{s\sim\sigma}{E}[\lambda\cdot\mathrm{cost}(s^*) + \mu\cdot\mathrm{cost}(s)]$$

$$= (1+\varepsilon)\lambda\cdot\mathrm{cost}(s^*) + (1+\varepsilon)\mu\cdot\underset{s\sim\sigma}{E}[\mathrm{cost}(s)]$$

$$\Rightarrow \underset{s\sim\sigma}{E}[\mathrm{cost}(s)] \le \frac{(1+\varepsilon)\lambda}{1-(1+\varepsilon)\mu}\cdot\mathrm{cost}(s^*)$$

Concrete example: Atomic routing, $(\frac{5}{3}, \frac{1}{3})$-smooth

$$\Rightarrow \frac{1}{\mu} - 1 = \frac{1}{\frac{1}{3}} - 1 = 2$$

$\Rightarrow$ if $\varepsilon < 2$, any ε-CCE is at most $\dfrac{(1+\varepsilon)\frac{5}{3}}{1-(1+\varepsilon)\frac{1}{3}} = \dfrac{5+5\varepsilon}{2-\varepsilon}$

worse than optimal

$\Rightarrow$ if $\varepsilon = 1$ (so deviating can halve your cost),

still ≤ 10 times worse than OPT!