2/24/22: Nonatomic Routing Games

Setup:

- Directed multigraph $G=(V,E)$
- source s, sink t
  (could have multiple source/sink pairs, results basically hold, see Roughgarden Exercise 11.5)
- rate $r>0$ (amount of traffic)
  (Tuesday: $r=1$)
- cost function $c_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ $\forall e \in E$
  - continuous, nonnegative, nondecreasing
  - affine: $c_e(x) = ax+b$, $a,b \geq 0$
- $P = \{s \to t \text{ paths}\}$ (set of strategies)
- flow: $\{f_p\}_{p \in P}$ s.t. $f_p \geq 0$ $\forall p \in P$, $\sum_{p \in P} f_p = r$
  - Given flow f, let $f_e = \sum_{p \in P : e \in p} f_p$
- $c_p(f) = \sum_{e \in p} c_e(f_e)$ = cost of using path p in f
- Cost of flow (social cost):

$$C(f) = \sum_{p \in P} f_p c_p(f)$$

$$= \sum_{P \in \mathcal{P}} f_P \sum_{e \in P} c_e(f_e)$$ (def of $c_P(f)$)

$$= \sum_{e \in E} \sum_{P \in \mathcal{P}: e \in P} f_P \, c_e(f_e)$$ (switch order of summation)

$$= \sum_{e \in E} c_e(f_e) \sum_{P \in \mathcal{P}: e \in P} f_P$$

$$= \sum_{e \in E} c_e(f_e) \cdot f_e$$ (def of f_e)

- Equilibrium flow (pure Nash):

$$c_P(f) \le c_{P'}(f) \quad \forall P, P' \in \mathcal{P}, \quad f_P > 0$$

$\Rightarrow$ all paths with nonzero flow have same cost, all other paths at least that cost

<u>Thm</u>: There is at least one equilibrium flow, and all equilibrium flows have the same cost

<u>Pf sketch</u>: (NRTV has details)

Potential, but continuous $\quad \Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(x)\,dx$

More complicated version of atomic routing argument:

any local minimum of $\Phi(f)$ is equilibrium flow,

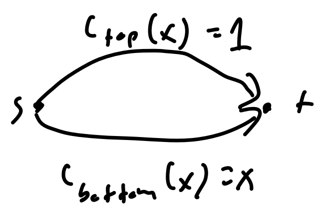
every equilibrium flow is a local min

Φ convex $\Rightarrow$ if f, f' equilibrium flows, then both global min, and chord between them all global min

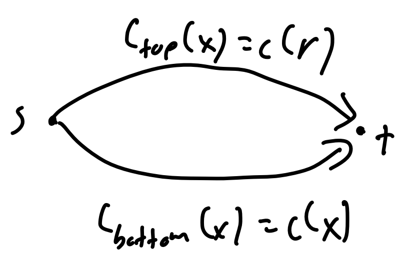
$\Rightarrow c_e(f_e) = c_e(f_e') \quad \forall e \in E$

<u>Pigou</u>:

Tuesday:



Today: fix function $c(x)$



If $c(x) = x$ and $r = 1$, just like Tuesday

Equilibrium flow:

All r on bottom since c nondecreasing

$\Rightarrow$ cost $r \cdot c(r)$

OPT: x on bottom, $r-x$ on top

$\Rightarrow$ cost $x \cdot c(x) + (r-x) \cdot c(r)$

$$\Rightarrow \text{PoA} = \sup_{0 \le x \le r} \left(\frac{r \cdot c(r)}{x \cdot c(x) + (r-x) \cdot c(r)} \right)$$

(so if $r=1$, $c(x)=x$ then sup achieved at $x=\frac{1}{2}$,
PoA $= \frac{4}{3}$)

Obs: $$\sup_{0 \le x \le r} \left(\frac{r \cdot c(r)}{x \cdot c(x) + (r-x) \cdot c(r)} \right) = \sup_{x \ge 0} \left(\frac{r \cdot c(r)}{x \cdot c(x) + (r-x) \cdot c(r)} \right)$$

(interpret $\frac{0}{0} = 1$)

Pf: If $x > r$,

$$\begin{aligned} x \cdot c(x) + (r-x) \cdot c(r) &= x \cdot c(x) + r \cdot c(r) - x \cdot c(r) \\ &\ge x \cdot c(r) + r \cdot c(r) - x \cdot c(r) \quad \text{(c non-decreasing)} \\ &= r \cdot c(r) \end{aligned}$$

$\Rightarrow$ denominator smallest in $[0, r]$

$$\Rightarrow \text{PoA} = \sup_{x \ge 0} \left(\frac{r \cdot c(r)}{x \cdot c(x) + (r-x) \cdot c(r)} \right)$$

Def: Let $\mathcal{C}$ be a class of cost functions (linear, quadratic, ...).

The Pigou bound for $\mathcal{C}$ is

$$\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left(\frac{r \cdot c(r)}{x \cdot c(x) + (r-x) \cdot c(r)} \right)$$

Obs: If $\mathcal{C}$ is affine functions, $\alpha(\mathcal{C}) = \frac{4}{3}$

If $\mathcal{C}$ is degree-p polynomials, $\alpha(\mathcal{C}) \approx \frac{p}{\ln p}$

Thm: Let $\mathcal{C}$ be a class of cost functions. Any nonatomic routing game with edge cost functions in $\mathcal{C}$ has price of anarchy $\leq \alpha(\mathcal{C})$

Pf: Let f^* optimal flow
f equilibrium flow $\Rightarrow$ WTS: $\frac{C(f)}{C(f^*)} \leq \alpha(\mathcal{C})$

Since $\alpha(\mathcal{C})$ includes $\sup_{r \geq 0} \sup_{x \geq 0}$, can choose any values for r, x to get a lower bound on $\alpha(\mathcal{C})$

Let $e \in E$. Use $c = c_e$, $r = f_e$, $x = f_e^*$ (so x could be $> r$):

$$\alpha(\mathcal{C}) \geq \frac{r \cdot c(r)}{x \cdot c(x) + (r-x) \cdot c(r)} = \frac{f_e \cdot c_e(f_e)}{f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*) \cdot c_e(f_e)}$$

$$\Rightarrow f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*) \cdot c_e(f_e) \geq \frac{1}{\alpha(\mathcal{C})} f_e \cdot c_e(f_e)$$

$$\Rightarrow \sum_{e\in E} \Big(\underbrace{f_e^* \cdot c_e(f_e^*)}_{C(f^*)} + (f_e - f_e^*)\cdot c_e(f_e) \Big) \geq \frac{1}{\alpha(\mathcal{C})} \underbrace{\sum_{e\in E} f_e \cdot c_e(f_e)}_{C(f)}$$

$$\Rightarrow C(f^*) + \sum_{e\in E} (f_e - f_e^*)\, c_e(f_e) \geq \frac{1}{\alpha(\mathcal{C})} \cdot C(f)$$

Almost what we want!

<u>WTS</u>: $\sum_{e\in E} (f_e - f_e^*)\, c_e(f_e) \leq 0$

$$\Leftrightarrow C(f) - \sum_{e\in E} f_e^* \, c_e(f_e) \leq 0$$

f an equilibrium flow $\Rightarrow$ all paths with $f_p > 0$ have same cost L

$$\Rightarrow C(f) = \sum_{p\in P} f_p \, c_p(f) = L \sum_{p\in P} f_p = rL$$

$$\sum_{e\in E} f_e^* \, c_e(f_e) = \sum_{e\in E} c_e(f_e) \sum_{p\in P : e\in p} f_p^* \quad \text{(def of } f_e^*\text{)}$$

$$= \sum_{p\in P} \sum_{e\in p} f_p^* \, c_e(f_e) \quad \text{(switch order of summation)}$$

$$= \sum_{p \in P} f_p^* \sum_{e \in p} c_e(f_e)$$

$$= \sum_{p \in P} f_p^* \, c_p(f)$$ (def of $c_p(f)$)

$$\geq L \sum_{p \in P} f_p^*$$ (f equilibrium flow)

$$= rL$$

$$\Rightarrow C(f) - \sum_{e \in E} f_e^* c_e(f_e) \leq 0$$ ✓

Application: Network Overprovisioning:

Suppose edge e has capacity u_e

Set $c_e(x) = \begin{cases} \frac{1}{u_e - x} & \text{if } x < u_e \\ \infty & \text{otherwise} \end{cases}$

(expected delay in M/M/1 queue)

Def: Network is β-overprovisioned if $f_e \leq (1-\beta) u_e$ for all $e \in E$ in an equilibrium flow

Proof of PoA thm only applies c_e functions to

f, f^*

$\Rightarrow$ always spare capacity

$\Rightarrow$ analyze $\alpha(\mathcal{C})$ under restriction β-spare capacity,

get $\text{PoA} \leq \frac{1}{2}\left(1 + \sqrt{\frac{1}{\beta}}\right)$

Pretty good! $\beta = \frac{1}{10}$, $\text{PoA} \leq 2.1$

$\Rightarrow$ in overprovisioned networks, selfish routing doesn't hurt us very much