601.436/636 Algorithmic Game Theory **Spring 2022**
Homework #5 **Due:** April 14, 2022, 4:30pm

Remember: you may work in groups of up to three people, but must write up your solution entirely on your own. Collaboration is limited to discussing the problems – you may not look at, compare, reuse, etc. any text from anyone else in the class. Please include your list of collaborators on the first page of your submission. You may use the internet to look up formulas, definitions, etc., but may not simply look up the answers online.

Please include proofs with all of your answers, unless stated otherwise. Your solution must by typeset (*not* handwritten), and must be submitted by gradescope.

---

1 Auctions with Distinct Goods (50 points)

Consider an auction setting with a set M of distinct goods. Each bidder i has a publicly known subset $T_i \subseteq M$ of goods that it wants, and a private valuation v_i of getting them. An allocation is a partition $(A_1, A_2, \ldots, A_n)$ of the goods (or, equivalently, a function from the goods to the bidders). The *social surplus* of an allocation is the sum over all bidders who receive their desired items of their valuations: if $(A_1, A_2, \ldots, A_n)$ is an allocation then the social surplus is $\sum_{i:T_i \subseteq A_i} v_i$.

(a) (16 points) Prove that this is a single-parameter environment.

(b) (17 points) Here is a natural greedy allocation rule, given a reported bid b_i from each player i:

  (a) Initialize $S = \emptyset$, $X = M$.
  (b) Sort and re-index the bidders so that $b_1 \geq b_2 \geq \cdots \geq b_n$.
  (c) For $i = 1, 2, 3, \ldots, n$:
    - If $T_i \subseteq X$, then:
      – Delete T_i from X.
      – Add i to S.
  (d) Return S (give the bidders in S their desired items)

  Does this algorithm define a monotone allocation rule? Prove it or give an explicit counterexample.

(c) (17 points) Prove that if all bidders report truthfully and have sets T_i of cardinality at most d, then the outcome of the allocation rule in (b) has social surplus at least $1/d$ times the social surplus of the optimal (surplus-maximizing) allocation.

2 First-Price Auctions (50 points)

In this problem we compare the revenue achieved by first- and second-price auctions for a single good. Analyzing what happens in a first-price auction is not trivial; the easiest way to proceed is to assume that each valuation v_i is drawn i.i.d. from a known prior distribution F. A strategy of a bidder i in a first price auction is then a predetermined formula for (under)bidding: formally, a function $b_i(\cdot)$ that maps its valuation v_i to a bid $b_i(v_i)$. You should conceptually think of this strategy (i.e., this function) as being announced to all of the other bidders in advance; but of course, the other bidders do not know the actual value of v_i (and hence do not know the corresponding bid $b_i(v_i)$). We will call such a family $b_1(\cdot), \ldots, b_n(\cdot)$ of bidding functions a *Bayes-Nash equilibrium* if for every bidder i and every valuation v_i, the bid $b_i(v_i)$ maximizes i's expected payoff, where the expectation is with respect to the random draws of the other bidders' valuations (which, via their bidding functions, induce a distribution over their bids).

(a) (25 points) Suppose each valuation is an independent draw from the uniform distribution on $[0, 1]$. Prove that one Bayes-Nash equilibrium is given by setting $b_i(v_i) = v_i(n-1)/n$ for every i and v_i.

(b) (25 points) Prove that the expected revenue of the seller at this equilibrium of the first-price auction is exactly the expected revenue of the seller with truthful bidding in a Vickrey (second-price) auction (where in both cases the expectation is over the valuation draws).