Remember: you may work in groups of up to three people, but must write up your solution entirely on your own. Collaboration is limited to discussing the problems – you may not look at, compare, reuse, etc. any text from anyone else in the class. Please include your list of collaborators on the first page of your submission. You may use the internet to look up formulas, definitions, etc., but may not simply look up the answers online.

Please include proofs with all of your answers, unless stated otherwise. Your solution must by typeset (*not* handwritten), and must be submitted by gradescope.

1 Upper Bound on PoA of Connection Games (50 points)

In class we showed an example of the connection game in which the price of anarchy of pure Nash equilibria was equal to k (the number of players). Prove that in any connection game, the price of anarchy of pure Nash equilibria is at most k. In other words, prove that any pure Nash has cost at most k times the optimal cost.

Solution: Consider some instance of the connection game with graph $G = (V, E)$, players $[k]$ with demands $(s_i, t_i)_{i \in [k]}$, and positive edge costs $\{c_e\}_{e \in E}$. Let $s = (P_1, P_2, \ldots, P_k)$ be some pure Nash equilibrium, and let $s^* = (P_1^*, P_2^*, \ldots, P_k^*)$ be the optimal solution. For every $i \in [k]$, let P_i' denote the shortest $s_i \to t_i$ path under edge lengths c. Then any player i could play P_i' as a strategy, and the most it would pay in that case would be $\sum_{e \in P_i'} c_e$ (which is what it would pay if no other player used any strategy which included any edge of P_i'). Since s is a pure Nash, we therefore get that $c_i(s) \leq \sum_{e \in P_i'} c_e$.

On the other hand, consider s^*. Clearly

$$c_i(s^*) = \sum_{e \in P_i^*} \frac{c_e}{h_e(s^*)} \geq \sum_{e \in P_i^*} \frac{c_e}{k} = \frac{1}{k} \sum_{e \in P_i^*} c_e \geq \frac{1}{k} \sum_{e \in P_i'} c_e,$$

where the first equality is by definition, the next inequality is because at most k players can use any edge, and the final inequality is because P_i' is a shortest path. Hence $c_i(s) \leq k \cdot c_i(s^*)$.

2 Location Game Matching Bound (50 points)

In class we proved that the location game (i.e. "competitive facility location with price-taking markets and profit-maximizing firms") is $(1, 1)$-smooth, and thus the the global value (i.e. social surplus) $V(s)$ of any coarse correlated equilibrium s is at least $1/2$ the global value of the optimal solution. Give an example which shows that this is tight in the strongest sense: in your example there should be a *pure Nash equilibrium* which has social surplus that is $1/2$ the social surplus of the optimal solution.

Solution: Consider an instance of the location game with $F = [3]$ (i.e., three locations), players in $[2]$ (i.e., two players), and $M = [2]$ (i.e., two markets). Set $v_1 = v_2 = 3$, so each market has valuation 3. Set $S_1 = \{1, 2\}$ and $S_2 = \{2, 3\}$. Set $c_{11} = c_{22} = 1$ (so it costs 1 to serve market 1 from location 1 and similarly costs 1 to serve market 2 from location 2), and set all other c_{ij} values to 3.

The optimal solution is the strategy vector $(1, 2)$, i.e., to have player 1 at location 1 and player 2 at location 2. We showed in class that the social surplus is equal to $\sum_{j \in M}(v_j - c_{f(j)j})$, where $f(j)$ is the location closest to market j with at least one player. Thus our optimal solution has social surplus $2 + 2 = 4$.

On the other hand, consider the strategy vector $(2, 3)$, i.e., the setting where player 1 is at location 2 and player 2 is at location 3. The social surplus here is equal to $(3-3)+(3-1) = 2$, which is half of the optimal social surplus, so we just need to show that this is a pure Nash. Note that in this strategy vector player 1 gets utility 2 (since it can charge market 2 a cost of 3 and only pay 1 to provide service, but gets no profit from market 1) and player 2 gets utility 0 (since it gets no profit from either market). If player 1 switches to its other strategy (location 1) then it will still have utility 2, since it will be able to make that profit from market 1 but will make no profit from market 2. Similarly, if player 2 switches to its other strategy (location 2) then it will be at the same location as player 1, so will continue to make 0 profit. Thus no player has incentive to switch, and hence this is a pure Nash.