601.436/636 Algorithmic Game Theory **Spring 2022**
Homework #3 **Due:** March 10, 2022, 4:30pm

Remember: you may work in groups of up to three people, but must write up your solution entirely on your own. Collaboration is limited to discussing the problems – you may not look at, compare, reuse, etc. any text from anyone else in the class. Please include your list of collaborators on the first page of your submission. You may use the internet to look up formulas, definitions, etc., but may not simply look up the answers online.

Please include proofs with all of your answers, unless stated otherwise. Your solution must by typeset (*not* handwritten), and must be submitted by gradescope.

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1 Multicommodity Nonatomic Routing (33 points)

(Roughgarden Exercise 11.5) Consider a *multicommodity* nonatomic routing instance $G = (V, E)$ where for each $i \in \{1, 2, \ldots, k\}$, there is r_i traffic (rate) from $s_i \in V$ to $t_i \in V$. So if $k = 1$ this is precisely the nonatomic setting considered in class.

(a) (8 points) Extend the definition of a *flow* and of an *equilibrium flow* to the multicommodity setting.

Solution:

- Let $\mathcal{P}_i$ denote the set of all $s_i \to t_i$ paths, and let $\mathcal{P} = \cup_{i=1}^{k} \mathcal{P}_i$. Then a flow is $\{f_P\}_{P \in \mathcal{P}}$ such that $f_P \geq 0$ for all $P \in \mathcal{P}$ and $\sum_{P \in \mathcal{P}_i} f_P = r_i$ for all $i \in [k]$. Given such a flow, let $f_e = \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i : e \in P} f_P$.
- Given a flow f, let $c_P(f) = \sum_{e \in P} c_e(f_e)$ for each path P (exactly as in class). A flow is an equilibrium flow if for all $i \in [k]$, $c_P(f) \leq c_{P'}(f)$ for all $P, P' \in \mathcal{P}_i$ with $f_P > 0$.

(b) (8 points) Extend the social cost objective from class (what Roughgarden calls the total travel time) to the multicommodity setting, and prove that (as in class) this can be written in two distinct ways (as a sum over paths and as a sum over edges).

Solution: We use the following definition of social cost in terms of paths and then rearrange terms to get the equivalent in terms of edge:

$$\begin{aligned}
C(f) &= \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_P c_P(f) \\
&= \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_P \sum_{e \in P} c_e(f_e) = \sum_{e \in E} \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i : e \in P} f_P c_e(f_e) = \sum_{e \in E} c_e(f_e) \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i : e \in P} f_P \\
&= \sum_{e \in E} c_e(f_e) f_e
\end{aligned}$$

(c) (17 points) Prove that Theorem 10.3.2 from class continues to hold in the multicommodity setting: the price of anarchy of multicommodity nonatomic routing with cost functions in $\mathcal{C}$ is at most $\alpha(\mathcal{C})$.

Solution: We follow the analysis from class. As there, let f denote an equilibrium flow and let f^* denote an optimal flow. For each edge e, instantiating the Pigou bound with $c = c_e$ and $r = f_e$ and $x = f_e^*$ yields that

$$\alpha(\mathcal{C}) \geq \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} = \frac{f_e \cdot c_e(f_e)}{f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*) \cdot c_e(f_e)}.$$

Rearranging and the summing over all $e \in E$ implies that

$$\begin{aligned} \sum_{e \in E} (f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*) \cdot c_e(f_e)) &\geq \frac{1}{\alpha(\mathcal{C})} \sum_{e \in E} f_e c_e(f_e) \\ \implies C(f^*) + \sum_{e \in E} (f_e - f_e^*) c_e(f_e) &\geq \frac{1}{\alpha(\mathcal{C})} C(f). \end{aligned}$$

So now we just need to show that $\sum_{e \in E} (f_e - f_e^*) c_e(f_e) \leq 0$, or equivalently that $C(f) - \sum_{e \in E} f_e^* c_e(f_e) \leq 0$. Everything we've done until now has been identical to class, but now we have to change things slightly. Since f is an equilibrium flow, we know that every path in $\mathcal{P}_i$ with nonzero flow in f must have the exact same cost; call this cost L_i. Hence

$$C(f) = \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_P c_P(f) = \sum_{i=1}^{k} L_i \sum_{P \in \mathcal{P}_i} f_P = \sum_{i=1}^{k} r_i L_i$$

On the other hand, we can analyze the second term:

$$\begin{aligned} \sum_{e \in E} f_e^* c_e(f_e) &= \sum_{e \in E} c_e(f_e) \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i : e \in P} f_P^* = \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} \sum_{e \in P} f_P^* c_e(f_e) = \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_P^* \sum_{e \in P} c_e(f_e) \\ &= \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_P^* c_P(f) \geq \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} f_P^* L_i = \sum_{i=1}^{k} L_i \sum_{P \in \mathcal{P}_i} f_P^* = \sum_{i=1}^{k} r_i L_i. \end{aligned}$$

Hence $C(f) - \sum_{e \in E} f_e^* c_e(f_e) \leq 0$ as required.

2 Nonatomic Routing with Fairness Objective (33 points)

(Roughgarden Problem 11.2) Suppose that instead of caring about total cost in nonatomic routing, our objective was the maximum cost: for a flow f, its cost is

$$C(f) = \max_{P \in \mathcal{P} : f_P > 0} c_P(f).$$

rather than the old $\sum_{P\in\mathcal{P}} f_P c_P(f)$. We're going to bound the Price of Anarchy with respect to this new cost function.

Suppose that all edges have affine cost functions, i.e., cost functions of the form $c_e(x) = a_e x + b_e$ for nonnegative a_e, b_e. For simplicity, assume that $r = 1$ (a flow sends one unit of traffic).

(a) (11 points) Suppose that G only has two vertices s and t and any number of parallel edges from s to t (each with their own affine cost function). Prove that the price of anarchy is 1.

Solution: Let f be an equilibrium flow and let f^* be the optimal flow with respect to the new cost function. By definition, $C(f^*) \leq C(f)$. Since every $s \to t$ path is just an edge (due to the structure of G), we can use $c_e(f_e) = a_e f_e + b_e$ to denote the cost of using the edge in f. Let $e' = \max_{e\in E: f_e>0} c_e(f_e)$ be the maximum cost edge used in f, i.e., $C(f) = c_{e'}(f_{e'})$. If $f^*_{e'} \geq f_{e'}$ then $C(f^*) \geq C(f)$, so we are finished. Otherwise $f^*_{e'} < f_{e'}$, in which case there must be some edge $e'' \in E$ such that $f^*_{e''} > f_{e''}$ since both f and f^* are flows of size 1. Thus

$$c_{e''}(f_{e''}) = a_e f_{e''} + b_e < a_e f^*_{e''} + b_e = c_{e''}(f^*) \leq C(f^*) \leq C(f) = c_{e'}(f_{e'}).$$

This contradicts the definition of an equilibrium flow: the players using edge e' could pay less by switching to e''. Hence $f^*_{e'} \leq f_{e'}$, and so $C(f^*) = C(f)$ and thus the price of anarchy is 1.

(b) (11 points) Prove that in general G, the Price of Anarchy can be at least 4/3.
Hint: remember Braess's paradox from Lecture 1.

Solution: Let's use the exact Braess's paradox example. Let f be the flow in which all the flow is sent along the zig-zag path. Then as in lecture 1, it's easy to see that in this flow the cost of the zig-zag path is 2, and the cost of all other paths is also 2, and hence it is an equilibrium flow since no player has incentive to deviate. In this equilibrium flow, the maximum path with nonzero flow has cost 2, and thus $C(f) = 2$.

Now consider the flow f^* in which 1/2 a unit of flow uses the top path and 1/2 a unit uses the bottom path. Then each of these paths has cost 3/2, and hence $C(f^*) = 3/2$.

(c) (11 points) Prove that in general G, the Price of Anarchy is at most 4/3.
Hint: you can use without proof the statement from class that the Pigou bound for affine cost functions is 4/3. Combine this with the main theorem from Lecture 10.

Solution: Let f be an equilibrium flow and let f^* be the optimum flow (with respect to our new cost function). Let C' denote the *old* cost function, i.e., $C'(f) = \sum_{P\in\mathcal{P}} f_P c_P(f) = \sum_{e\in E} f_e c_e(f_e)$. Since f is an equilibrium flow, we know that every path with nonzero flow has the exact same cost in f. Hence $C'(f) = \sum_{P\in\mathcal{P}} f_P c_P(f) = r \cdot C(f) = C(f)$, since every path in f must have the same cost as the maximum cost path being used, which has cost $C(f)$ by definition of C. On the other hand, we know that

$$C'(f^*) = \sum_{P \in \mathcal{P}} f_P^* c_P(f^*) \leq \max_{P \in \mathcal{P}: f_P^* > 0} c_P(f^*) = C(f^*),$$

since $\sum_{P \in \mathcal{P}} f_P^* = 1$. And we know from class that $C'(f) \leq \frac{4}{3} C'(f^*)$. Putting this all together, we get that

$$C(f) = C'(f) \leq \frac{4}{3} C'(f^*) \leq \frac{4}{3} C(f^*).$$

Thus the price of anarchy is at most 4/3.

3 Atomic Routing Games (34 points)

(NRTV Exercise 18.3a) An *asymmetric scheduling instance* differs from an atomic routing instance in the following two respects. First, the underlying network is restricted to a common source vertex s, a common sink vertex t, and a set of parallel links that connect s to t. On the other hand, we allow different players to possess different strategy sets: each player i has a prescribed subset S_i of the links that it is permitted to use.

Show that every asymmetric scheduling instance is equivalent to an atomic routing game. Your reduction should make use only of the cost functions of the original scheduling instance, plus possibly the all-zero cost function.

Solution: Given an asymmetric scheduling instance with edge set E, cost functions $c_e : \mathbb{R} \to \mathbb{R}$ for each $e \in E$, and subsets $S_i \subseteq E$ for each player $i \in [k]$, we construct an atomic routing game as follows. First, we create a new directed graph $G' = (V', E')$. The vertex set V' will consist of a common sink t', a source vertex s_i for each player $i \in [k]$, and a vertex v_e for each edge $e \in E$ in the original graph. We now define E' and the cost function of the edges in E'. For each $e \in E$, we add an edge (v_e, t') to E' with cost function $c_{(v_e,t')} = c_e$. For each $i \in [k]$ and $e \in S_i$, we add an edge (s_i, v_e) to E' with cost function 0. We now have a complete instance of atomic routing: the graph is $G' = (V', E')$, the edge costs are as defined, there are k players, and for each player i the source is s_i and the sink is t'. We claim that this is equivalent to the original asymmetric scheduling instance.

Let's first consider the strategy sets. Let $\mathcal{P}_i$ denote the strategy set of player i in the atomic routing instance, i.e., the set of all s_i to t' paths. We claim that there is a bijection between $\mathcal{P}_i$ and S_i. Every path in $\mathcal{P}_i$ is of the form $s_i \to v_e \to t'$, where $e \in S_i$. And similarly, for every $e \in S_i$ there is a path from s_i to t' of the form $s_i \to v_e \to t'$. Thus there is a bijection between $\mathcal{P}_i$ and S_i. Let $\pi_i : \mathcal{P}_i \to S_i$ denote this bijection.

Let $f = (e_1, \ldots, e_k) \in S_1 \times \cdots \times S_k$. For each $e \in E$, let $f_e = |\{i : e = e_i\}|$. Then for each $i \in [k]$, the cost to player i of f in the asymmetric scheduling instance is precisely $c_{e_i}(f_{e_i})$ and the total cost is $\sum_{i=1}^k c_{e_i}(f_{e_i})$. Let $\hat{f} = (\pi_1^{-1}(e_1), \pi_2^{-1}(e_2), \ldots, \pi_k^{-1}(e_k))$ be the corresponding strategy vector in the atomic routing game. Then since the cost of every edge (s_i, e_i) in E' is 0, the cost to player i of $\hat{f}$ is $0 + c_{e_i}(f_{e_i}) = c_{e_i}(f_{e_i})$ and the total cost is also $\sum_{i=1}^k c_{e_i}(f_{e_i})$.

The same argument works to show that given a strategy vector $(P_1, \ldots, P_k) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_k$, the cost to every player and the total cost in the atomic routing game is the same as the cost to every player and the total cost of $(\pi_1(P_1), \ldots, \pi_k(P_k))$ in the asymmetric scheduling instance. Thus the atomic routing game is equivalent to the asymmetric scheduling instance.