601.436/636 Algorithmic Game Theory **Spring 2022**
Homework #3 **Due:** March 10, 2022, 4:30pm

Remember: you may work in groups of up to three people, but must write up your solution entirely on your own. Collaboration is limited to discussing the problems – you may not look at, compare, reuse, etc. any text from anyone else in the class. Please include your list of collaborators on the first page of your submission. You may use the internet to look up formulas, definitions, etc., but may not simply look up the answers online.

Please include proofs with all of your answers, unless stated otherwise. Your solution must by typeset (*not* handwritten), and must be submitted by gradescope.

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1 Multicommodity Nonatomic Routing (33 points)

(Roughgarden Exercise 11.5) Consider a *multicommodity* nonatomic routing instance $G = (V, E)$ where for each $i \in \{1, 2, \ldots, k\}$, there is r_i traffic (rate) from $s_i \in V$ to $t_i \in V$. So if $k = 1$ this is precisely the nonatomic setting considered in class.

(a) (8 points) Extend the definition of a *flow* and of an *equilibrium flow* to the multicommodity setting.

(b) (8 points) Extend the social cost objective from class (what Roughgarden calls the total travel time) to the multicommodity setting, and prove that (as in class) this can be written in two distinct ways (as a sum over paths and as a sum over edges).

(c) (17 points) Prove that Theorem 10.3.2 from class continues to hold in the multicommodity setting: the price of anarchy of multicommodity nonatomic routing with cost functions in $\mathcal{C}$ is at most $\alpha(\mathcal{C})$.

2 Nonatomic Routing with Fairness Objective (33 points)

(Roughgarden Problem 11.2) Suppose that instead of caring about total cost in nonatomic routing, our objective was the maximum cost: for a flow f, its cost is

$$C(f) = \max_{P \in \mathcal{P}: f_P > 0} c_P(f).$$

rather than the old $\sum_{P \in \mathcal{P}} f_P c_P(f)$. We're going to bound the Price of Anarchy with respect to this new cost function.

Suppose that all edges have affine cost functions, i.e., cost functions of the form $c_e(x) = a_e x + b_e$ for nonnegative a_e, b_e. For simplicity, assume that $r = 1$ (a flow sends one unit of traffic).

(a) (11 points) Suppose that G only has two vertices s and t and any number of parallel edges from s to t (each with their own affine cost function). Prove that the price of anarchy is 1.

(b) (11 points) Prove that in general G, the Price of Anarchy can be at least $4/3$.
Hint: remember Braess's paradox from Lecture 1.

(c) (11 points) Prove that in general G, the Price of Anarchy is at most $4/3$.
Hint: you can use without proof the statement from class that the Pigou bound for affine cost functions is $4/3$. Combine this with the main theorem from Lecture 10.

3 Atomic Routing Games (34 points)

(NRTV Exercise 18.3a) An *asymmetric scheduling instance* differs from an atomic routing instance in the following two respects. First, the underlying network is restricted to a common source vertex s, a common sink vertex t, and a set of parallel links that connect s to t. On the other hand, we allow different players to possess different strategy sets: each player i has a prescribed subset S_i of the links that it is permitted to use.

Show that every asymmetric scheduling instance is equivalent to an atomic routing game. Your reduction should make use only of the cost functions of the original scheduling instance, plus possibly the all-zero cost function.