1 Potential Games (50 points)

A team game is a game in which all players have the same utility function: \( u_1(s) = \cdots = u_n(s) \) for every strategy profile \( s \). In a dummy game, the utility of every player \( i \) is independent of its strategy: \( u_i(s) = u_i(s_{\sim i}, s'_i) \) for every \( s \in S \) and \( s'_i \in S_i \).

Prove that a game with utilities \( u_1, \ldots, u_n \) is a potential game (i.e., admits a potential function \( \Phi \)) if and only if it is the sum of a team game \( u'_1, \ldots, u'_n \) and a dummy game \( u^d_1, \ldots, u^d_n \) (i.e., \( u_i(s) = u'_i(s) + u^d_i(s) \) for all \( i \) and \( s \)).

2 No-Regret and Coarse Correlated Equilibria (50 points)

We proved that if every player uses a no-regret algorithm, then the time-averaged distribution converges to a coarse correlated equilibrium. Let’s prove almost the converse. Consider a \( k \)-player cost-minimization game in which \( C_i(s) \neq C_i(s') \) for every agent \( i \) and every pair of strategy profiles \( s, s' \in S \), (i.e., no agent has the same cost for two different profiles). Let \( \sigma \) be a coarse correlated equilibrium for this game. Prove that there exist no-regret algorithms \( A_1, A_2, \ldots, A_k \) for the agents so that the time-averaged history of the corresponding no-regret dynamics converges to \( \sigma \) as \( T \) tends to infinity.

Hint: Pre-program \( \sigma \) into \( A_1, \ldots, A_k \). To make sure that each algorithm \( A_i \) is a no-regret algorithm, switch to your favorite no-regret algorithm (e.g., multiplicative weights) if some other agent \( j \) fails to use the agreed-upon algorithm \( A_j \).