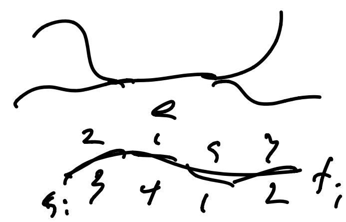
Connection Game (Network Formation)

- Directed graph $G=(V,E)$ - Edge costs $c_e \in \mathbb{R}_{\geq 0}$ $\forall e \in E$
- k players, player i has $(s_i, t_i) \in V \times V$
- Player i has strategies $P_i = \{ s_i \to t_i \text{ paths} \}$

$$S = P_1 \times P_2 \times \cdots \times P_k$$

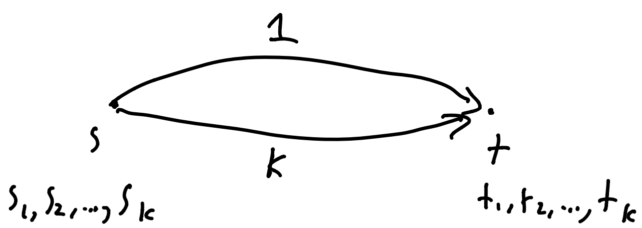
- Let: $s = (P_1, P_2, \ldots, P_k) \in S$

$$k_e(s) = |\{ i : e \in P_i \}| \quad \forall e \in E$$



- Player costs $C_i(s) = \sum_{e \in P_i} \frac{c_e}{k_e(s)}$
- Global cost $\text{cost}(s) = \sum_{i=1}^{k} C_i(s) = \sum_{e \in \bigcup_{i=1}^{k} P_i} c_e$

Price of Anarchy:



OPT: 1 (all top)

All bottom: cost k

Nash: $C_i(s) = 1$

switched: $\frac{1}{\ }$

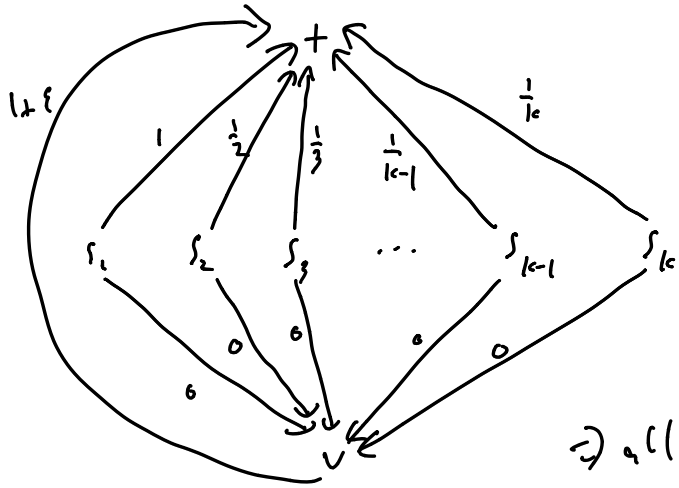
$\Rightarrow$ PoA $= k$

$$\text{PoS} = \underline{1} = \frac{\text{Best Nash}}{\text{OPT}}$$

Price of Stability:

Claim: Price of Stability $\geq \frac{H_k}{1+\varepsilon} = \frac{\sum_{i=1}^{k} \frac{1}{i}}{1+\varepsilon} = \Theta(\ln k)$

$\forall \varepsilon > 0$



OPT: $1+\varepsilon$ (all 2-hop)

Sps $A \subseteq [k]$ use 2-hop path

Let i max in A

$\Rightarrow C_i(s) = \frac{1+\varepsilon}{|A|} \geq \frac{1+\varepsilon}{i}$

$\Rightarrow$ i wants to deviate

$\Rightarrow$ all 1-hop only Nash

$\text{cost(Nash)} = \sum_{i=1}^{k} \frac{1}{i} = H_k$

Thm: The Price of Stability in any connection game is $\leq H_k$

First: Show potential game

$$\Psi_e(s) = c_e \cdot H_{k_e(s)} \quad \forall e \in E \qquad \Psi(s) = \sum_{e \in E} \Psi_e(s)$$

$(H_0 = 0)$

Lemma: Let $s = (P_1, \ldots, P_k) \in S$ and let $P_i' \in \mathcal{P}_i$. Then

$$\Psi(s) - \Psi(s_{-i}, P_i') = C_i(s) - C_i(s_{-i}, P_i')$$

Pf:

$$\sum_{e \in E} \Psi_e(s) - \sum_{e \in E} \Psi_e(s_{-i}, P_i') =$$

$$\begin{aligned}
\Psi(s) - \Psi(s_{-i}, P_i') &= \sum_{e \in E} \big(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\big) \\
&= \sum_{e \in P_i \setminus P_i'} \big(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\big) + \sum_{e \in P_i' \setminus P_i} \big(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\big) \\
&\quad + \cancel{\sum_{e \in P_i \cap P_i'} \big(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\big)} + \cancel{\sum_{e \in E \setminus (P_i \cup P_i')} \big(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\big)} \\
&= \sum_{e \in P_i \setminus P_i'} \big(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\big) + \sum_{e \in P_i' \setminus P_i} \big(\Psi_e(s) - \Psi_e(s_{-i}, P_i')\big) \\
&= \sum_{e \in P_i \setminus P_i'} \big(c_e H_{k_e(s)} - c_e H_{k_e(s)-1}\big) + \sum_{e \in P_i' \setminus P_i} \big(c_e H_{k_e(s)} - c_e H_{k_e(s)+1}\big) \\
&= \sum_{e \in P_i \setminus P_i'} \frac{c_e}{k_e(s)} - \sum_{e \in P_i' \setminus P_i} \frac{c_e}{k_e(s)+1} \\
&= \sum_{e \in P_i \setminus P_i'} \frac{c_e}{k_e(s)} - \sum_{e \in P_i' \setminus P_i} \frac{c_e}{k_e(s_{-i}, P_i')} + \sum_{e \in P_i \cap P_i'} \left(\frac{c_e}{k_e(s)} - \frac{c_e}{k_e(s_{-i}, P_i')}\right) \\
&= \sum_{e \in P_i} \frac{c_e}{k_e(s)} - \sum_{e \in P_i'} \frac{c_e}{k_e(s_{-i}, P_i')} \\
&= C_i(s) - C_i(s_{-i}, P_i')
\end{aligned}$$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k_e(s)}$$
$$-\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k_e(s)-1}\right)$$

✓

Side note (important later): lemma holds even if $P_i' = \emptyset$

<u>Lemma</u>: $\text{cost}(s) \le \Psi(s) \le H_k \cdot \text{cost}(s) \quad \forall s \in S$

<u>Pf</u>: $s = (P_1, P_2, \dots, P_k)$

$$\text{cost}(s) = \sum_{e \in \bigcup_{i=1}^{k} P_i} c_e \le \sum_{e \in E} c_e H_{k_e(s)} = \Psi(s)$$

$H_{k_e(s)} \le H_k$

$$= \sum_{e \in \bigcup_{i=1}^{k} P_i} c_e H_{k_e(s)} \le H_k \sum_{e \in \bigcup_{i=1}^{k} P_i} c_e = H_k \cdot \text{cost}(s)$$

<u>Proof of Theorem</u>:

Let s = global minimizer of Ψ (Pure Nash)

$s^* = OPT$

$\Rightarrow \text{cost}(s) \le \Psi(s) \le \Psi(s^*) \le H_k \cdot \text{cost}(s^*)$

Strong Nash:

Def: Let $s \in S$. The strategies $s'_A \in \prod_{i \in A} S_i$ are a **beneficial deviation** for $A \subseteq [k]$ if

P_i

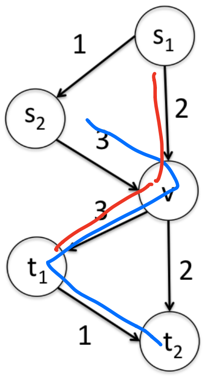
$c_i(s_{-A}, s'_A) \leq c_i(s)$ $\forall i \in A$, and $\exists$ $i \in A$ s.t.

$c_i(s_{-A}, s'_A) < c_i(s)$

Def: $s \in S$ is a **strong Nash** if there are no beneficial deviations for any $A \subseteq [k]$

Note: Every strong Nash is pure Nash

Don't always exist, even in connection game!



- Both 2-hop path: cost 5 each

  deviate together: cost 4 each

- Both 3-hop path: player 1 deviate

- One 3-hop, one 2-hop: player with 3-hop deviate 2-hop

<u>Thm</u>: If $s_i = s$ and $t_i = t$ $\forall i \in [k]$, then $\exists$ a strong Nash

<u>Pf</u>: shortest path

Thm: In a connection game, every strong Nash s has $\text{cost}(s) \le H_k \cdot \text{cost}(s^*)$ $\forall s^* \in S$

Pf: $s = (P_1, \dots, P_k)$ $\quad s^* = (P_1^*, P_2^*, \dots, P_k^*)$

Since s SN, no coalition deviates

In particular: $A_k = [k]$ does not deviate to s^*

some player i_k s.t. $C_{i_k}(s) \le C_{i_k}(s^*)$

$A_{k-1} = A_k \setminus \{i_k\}$

$\exists\, i_{k-1} \in A_{k-1}$ s.t. $C_{i_{k-1}}(s) \le C_{i_{k-1}}(s_{-A_{k-1}}, s^*_{A_{k-1}})$

$A_j = A_{j+1} \setminus \{i_{j+1}\}$

$C_{i_j}(s) \le C_{i_j}(s_{-A_j}, s^*_{A_j})$

$$\text{cost}(s) = \sum_{j=1}^{k} C_{i_j}(s) \le \sum_{j=1}^{k} C_{i_j}(s_{-A_j}, s^*_{A_j})$$

$$\le \sum_{j=1}^{k} C_{i_j}(s^*_{A_j}) = \sum_{j=1}^{k} \left(C_{i_j}(s^*_{A_j}) - C_{i_j}(s^*_{A_j - i_j}, \emptyset) \right)$$

$$= \sum_{j=1}^{k} \left(\Psi(S^*_{A_j}) - \Psi(S^*_{A_{j-1}}) \right)$$

$$= \Psi(S^*) - \Psi(S^*_{\emptyset}) = \Psi(S^*)$$

$$\leq H_k \cdot \mathrm{cost}(S^*)$$