Distributed Minimum Degree Spanning Trees

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**Input:** Graph $G = (V, E)$

**Feasible Solution:** Spanning tree $T = (V, E')$

**Objective:** Minimize max degree $\Delta(T)$ of $T$
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**Objective:** Minimize max degree \( \Delta(T) \) of \( T \)

- Classical NP-hard problem (reduction from Hamiltonian Path)
- Natural in distributed/networked settings:
  - Building a low-degree backbone network
  - Broadcast capacity in mobile telephone model [D, Halldórsson, Newport, Weaver DISC ’19]
  - In many networking scenarios, degree \( \approx \) load. Min max load.
  - Lots of attention to MST problem – why not MDST?
### Algorithms for MDST

- **Centralized (d = OPT):**
  - [Fürer, Raghavachari SODA '92]: Polytime $(d + 1)$-solution via complex recursive local search (*semi-local improvements*)
  - Simpler version (non-recursive) gives $(2d + \log n)$-solution (*local improvements*)

- **Distributed:**
  - [Blin, Butelle IPDPS '03]: Each local and semi-local improvement of FR can be computed in a distributed way. But separate improvements not computed in parallel, so still large running time ($\Omega(n)$).
  - Self-stabilizing algorithms [Blin, Fraigniaud ICDCS '15], [Blin, Potop-Butucaru, Rovedakis ’11]: running times $\Omega(n^2)$.

- More general problem: find MST which minimizes max degree. [Lavault, Valencia-Popon '08]: Still $\Omega(n^2)$.

- **Question:** Can we provide good approximations for MDST in time $O(D + \sqrt{n})$ (like for MST)?
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Our Results: Upper Bounds

Models:

- CONGEST: synchronous rounds, each message size at most $O(\log n)$ bits ($O(1)$ words)
- broadcast-CONGEST: in each round, each node sends same message to all neighbors

Theorem

There is a randomized algorithm in the broadcast-CONGEST model which builds a spanning tree of maximum degree at most $4(1+\epsilon)d + O(\log n)$ and has expected running time at most $O((D + \sqrt{n}) \log 4n)$.

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There is a deterministic algorithm in the CONGEST model which builds a spanning tree of maximum degree at most $4(1+\epsilon)d + O(\log n)$ and has expected running time at most $O((D + \sqrt{n}) \log 5n)$.
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- Want to show that polynomial dependence on n is necessary
- Precise lower bound rather complex. Some simple corollaries:

**Theorem**

For any $\epsilon < 1/6$, there exists a family of instances of diameter $D = \Theta(n^{1/2-3\epsilon} + \log n)$ where any MDST algorithm with a polylogarithmic multiplicative approximation factor needs $\tilde{\Omega}(n^{1/2-\epsilon} + D)$ rounds.

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Today: only upper bounds
Local improvements

Idea: Adding an edge creates a cycle. Can remove any edge in that cycle.

- Degrees of $x, y$ increase by 1, but degree of $u$ decreases by 1.
- FR: Make local improvements to decrease degrees of high-degree nodes, until no such local improvements exist.
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Def: Let $X_k$ be nodes of degree at least $k$ in current tree $T$.

Two parameters
- $\gamma$: definition of “high-degree” (nodes in $X_\gamma$)
- $q$: definition of “low-degree” (degree at most $\gamma_0 := \gamma - 2q$)

Local improvements:
- Hurt low-degree nodes by at most $q$
- Improve high-degree nodes, but by at most $q$ (so still worse than low-degree nodes)
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- Fix $q$, find $\gamma$ which gives “large” improvement. Repeat until no such $\gamma$.
- Try all $q$
- Complicated potential function to prove $\tilde{O}(1)$ iterations
- Approximation basically from centralized analysis (some extra loss)
Everything else

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**Main point (rest of talk):** want to find lots of parallel local improvements.
Leaf Branches

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Edge is *good* if:

- Both endpoints degree at most $\gamma_0$
- Endpoints in different branches, one of which is a leaf branch

We'll only make improvements from adding good edges.
Improvement Graph

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Create new bipartite graph: *improvement graph*

- $\mathcal{U}$: leaf branches
- $\mathcal{Q}$: nodes of degree $\leq \gamma_0$
- Edges: good edges in $G$

Approach: Find a large $(1, q)$-matching in improvement graph (degree 1 on leaf branches, degree $\leq q$ on low-degree nodes).
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What about requirement that high-degree edges improve at most $q$?
Constrained $q$-Matching

High-degree node improves $\geq q$ only if $\geq q$ leaf branches adjacent to it are matched.

\[ \text{Partition } U \text{ into bundles which share a parent. Require each bundle to only have } q \text{ matched nodes.} \]

So want to prove:

There must exist a large Constrained $q$-Matching (if $\gamma$ large enough and not many “medium-degree” nodes)

Can quickly find $O(1)$-approximation to Max Constrained $q$-Matching.
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- There must exist a large Constrained $q$-Matching (if $\gamma$ large enough and not many “medium-degree” nodes)
- Can quickly find $O(1)$-approximation to Max Constrained $q$-Matching.
There is a constrained $q$-matching of size at least $\frac{q}{\gamma} ((\gamma - 2)|X_\gamma| - d|X_{\gamma_0}|)$ (where $d$ is optimal max degree)
Large Constrained $q$-Matching

**Theorem**

There is a constrained $q$-matching of size at least $\frac{q}{\gamma} \left( (\gamma - 2)|X_\gamma| - d|X_{\gamma_0}| \right)$ (where $d$ is optimal max degree)

**Proof Sketch:**

- Each node in $X_\gamma$ causes lots of leaf branches when removed.
- **OPT** connects these components without using many edges.
- Counting and averaging.
Large Constrained $q$-Matching

**Theorem**

*There is a constrained $q$-matching of size at least $\frac{q}{\gamma}((\gamma - 2)|X_\gamma| - d|X_{\gamma_0}|)$ (where $d$ is optimal max degree)*

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So if $|X_{\gamma_0}|$ not much larger than $X_\gamma$ and $\gamma$ large enough, about $q|X_\gamma|$
Approximating Constrained $q$-Matching

Turn constrained $q$-matching into flow:

- Integral capacities, so integral max flow
- Flow of $\alpha$ iff constrained $q$-matching of size $\alpha$
Lemma: In depth-$d$ flow network, any maximal flow has value at least $(1/d)$ of max flow.

- High-Level Algorithm:
  - Start each flow path with $1/m$ flow
  - In each round: double flow in each flow path, unless some node on path already “full”
Approximate Max Flow

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**High-Level Algorithm:**
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- Gives approximate fractional maximal flow.
- Randomized rounding to get approximate constrained $q$-matching
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- Gives approximate fractional maximal flow.
- Randomized rounding to get approximate constrained $q$-matching
- Some annoying details ($U$ are actually components, not nodes)
Want to find many parallel local improvements
Restrict to “safe” improvements: add good edge, remove edge from leaf branch
Prove there have to be many safe improvements (or else finished)
Algorithm for approximating max safe improvements via max flow, randomized rounding.
$\tilde{O}(D + \sqrt{n})$ rounds using complex but standard CONGEST communication ideas.
Putting it Together

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Approximation guarantee:
- Somewhat complex/delicate
- Morally: same as centralized FR, but with small extra loss (2 to 4 multiplicative)
Results:

- Gave first $\tilde{O}(D + \sqrt{n})$-round $O(d + \log n)$-degree bounded MDST algorithm in CONGEST.
- Lower bound: Can’t really improve running time even if only want $O(d \cdot \log n)$-degree.
Conclusion

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Open Problems:
- Key idea was finding many local improvements which can be done in parallel. Should imply PRAM, streaming, etc. algorithms?
- Stronger approximation! Ideally: degree $d + 1$?
- Steiner tree instead of spanning?
  - Centralized bounds same (degree $d + 1$ from local search)
  - Centralized algorithm makes local improvements using paths instead of edges. Distributed?
Thanks!