8.1 Introduction

In this lecture we'll talk about a useful abstraction, priority queues, which are usually implemented through data structures known as heaps. In fact, priority queues are so closely identified with heaps that the terms are sometimes used interchangeably.

In a normal, queue, we insert elements (known as a push) and remove elements (known as a pop). Because it's a queue, pops happen in first-in-first-out (FIFO) order. This means that the obvious data structure is a linked list. In a *priority* queue, the order of pops is by priority, rather than in FIFO order. That is, we think of inserting elements which have keys which have a total order – for concreteness, let's set all keys to be integers. And since we don't actually care about the data in the elements, we'll just think of inserting integers. Then the intuition is that we should always pop off the smallest remaining key, no matter when it was inserted.

Slightly more formally, we want to support the following operations:

1. Insert(H, x): insert element x into the heap H
2. Extract-Min(H): remove and return an element with smallest key
3. Decrease-Key(H, x, k): decrease the key of x to k.

The following operations are also sometimes useful:

1. Find-Min(H): return the element with smallest key
2. Delete(H, x): delete element x from heap H
3. Meld(H_1, H_2): replace heaps H_1 and H_2 with their union

What if we try to do this with just a linked list? Insert is $O(1)$, Meld is $O(1)$, and depending on exactly how its implemented (if we assume the user has pointers to individual elements) Insert and Decrease-Key might take $O(1)$ time. But what about Extract-Min? We would need to look through the entire list, so we only get a running time bound of $O(n)$.

It turns out that heaps can be made extremely efficient. The best known bounds are for a structure known as *Fibonacci heaps*, but we won't have time to discuss them in detail. Instead, we'll first discuss *binary heaps*, which let us decrease the cost of Extract-Min at the cost of increasing some the cost of some other operations.

8.2 Binary Heaps

A binary heap is essentially just a complete binary tree, where the only nodes that can be missing are on the bottom level. We make sure that at the bottom level, nodes are filled in from "left" to "right". The only additional requirement is that the nodes are in *heap order*: the key of any node is no larger than the key of its children. So as we move up the tree keys are non-increasing, but in different branches the keys are not particularly related. Note that heap order implies that the minimum node is at the root, since if it were anywhere else it would have a parent which was larger than it.

Note that since binary heaps are essentially complete binary trees, we know that their height is at most $\log n$. This is a great property to have, since often the running time of an operation will depend on the height.

Let's see an example heap:

The basic operations are all reasonably straightforward. We'll assume that the data structure H itself contains a pointer to the root and to the last node (the rightmost node of the bottom level). Every node has a pointer to its left child and to its right child, as well as a pointer to its parent.

- Insert(H, x): we begin by inserting x into the next open spot (i.e. if the bottom level is not filled then the next location in it, and if the bottom level is full we start a new level). But this might violate heap order: x might be smaller than its parent. So we simply "swim up" – as long as x is smaller than its parent, we switch it and its parent. Clearly this takes time $O(\log n)$ in the worst case (and also amortized).

- Extract-Min(H): we know because of heap ordering that the minimum as at the root. So Find-Min(H) is easy: simply return the root, which takes $O(1)$ time. But to do Extract-Min we need to also remove this element. To do this, we first switch it with the final element (violating heap order). Now the minimum element has no children, so we can just return it and delete it. But we've violated heap order, so we need to fix the tree. We do this by letting the root "sink down" – as long as it is larger than one of its two children, we swap it with the smaller of its children. It is easy to see that at the end of this process we have restored heap order. And since the depth of the tree is $O(\log n)$, this only takes $O(\log n)$ time. It turns out we can actually also get $O(1)$-amortized.

- Decrease-Key(H, x, k): we can again just decrease the key of x, and have it swim up until we restore heap order. This takes $O(\log n)$ time.

- Delete(H, x): just like Extract-Min. We can swap x with the last node, remove x, and then restore heap order by sinking down.

- Meld(H_1, H_2). This is a more interesting operation. Let's assume that both heaps have size n. One option would be to simply iterate of all elements of H_2 and insert them into H_1. Since insert takes time $O(\log n)$, this would take time $O(n \log n)$.

  It turns out that we can do better. The first solution inserted elements one at a time and let them swim up. Let's try the opposite: we'll insert all elements at once and then swim

down. In $O(n)$ time we can iterate through the elements of H_2, inserting each of them at next open spot in H_1. So now we might have really drastically violated heap order, since we did n insertions without any repairing. But we've only used up $O(n)$ time. Now we iterate through the heap in *backwards order*. When considering node x (say at position i), we sink it down and then continue. Note that this maintains the invariant that the subtree rooted at i is in heap order.

So for nodes in the bottom level (which are the first we consider), since they have no children we don't swap them with anything. Now at the next level, we check if each node is larger than the smaller of its two children, and if so we swap them. We continue this until we've fixed the whole heap.

To analyze the running time, consider what happens when we're looking at a node x that's in level h (where we say that the bottom level is level 0, so the level is the height). We might have to sink x all the way to the bottom, which takes time h. But there are at most $\lceil n/2^{h+1} \rceil$ nodes at level h. So the total cost is only

$$\sum_{h=0}^{\log n} h \left\lceil \frac{n}{2^{h+1}} \right\rceil \leq n \sum_{h=0}^{\log n} \frac{h}{2^h} \leq O(n)$$

Amortized Extract-Min: Let's now look at Extract-Min from an amortized point of view (this will also work for delete). For a node x at depth d, define the weight of x to be $w(x) = d$. So the root has weight 0, each of its children has weight 1, each of their children have weight 2, etc. We will use as a potential function the sum of these weights, so $\Phi = \sum_x w(x)$.

Let's start by showing that we don't hurt insert (or any operation that causes a swim up). Let d be the depth that we're inserting at. Than an insert takes d time to swim up, but we have the added cost of the potential difference. But now there is just one more node at depth d, so the total amortized cost is at most $d + \Delta\Phi \leq 2d = O(d)$. Since d is $O(\log n)$, the amortized cost of inserting is still $O(\log n)$.

Now let's look at Extract-Min. The amount of time necessary to swim down after we do the swap is the depth d. On the other hand, what is $\Delta\Phi$? There is now one less node of depth d, so Φ decreased by d. So the whole cost of swimming down is paid for by the potential function, and the amortized cost is just $O(1)$ (for the initial swap).

8.3 Binomial Heaps

Binary heaps are all well and good, but for some uses the high cost of Meld makes them impractical. If we're doing a lot of Melds, then even the $O(n)$ bound isn't good enough. Can we make it more like $O(\log n)$? It turns out that we can, by using a data structure known as a *binomial heap*.

There are a few key ideas to binomial heaps, but first we should define a binomial tree (rather than a binary tree). We do this inductively. The binomial tree of order 0, which we call B_0, is just a single node. The binomial tree of order k, which we will call B_k is simply one B_{k-1} linked to another B_{k-1}. Let's do a few small cases:

There are a few simple properties of binomial trees which we will use. All can be proved easily by induction on k.

Lemma 8.3.1 *The order k binomial tree B_k has the following properties:*

1. *Its height is k.*
2. *It has 2^k nodes*
3. *The degree of the root is k*
4. *If we delete the root, we get k bionomial trees $B_{k-1}, \ldots, B_0$.*

Now that we've defined binomial trees, we can define a binomial heap.

Definition 8.3.2 *A binomial heap is a collection of binomial trees so that each tree is heap ordered, and there is exactly 0 or 1 tree of order k for each integer k.*

There are a few extra pointers that we need to keep around to make everything efficient. In particular, we'll keep the roots of the trees in a linked list, from smallest order to largest.

Like we've seen with the simple dictionary, there's a nice correspondence here to binary counters. In particular, suppose that we have n items total. Then since B_k must have size exactly 2^k if it exists, we know exactly which binomial trees are present and which are not! We don't know what's in each tree, but the k's where B_k exists must exactly correspond to the bits that are 1 in the binary representation of n.

This implies that there are at most $\log n$ trees in the heap, each of which has height at most $O(\log n)$. Together with the heap ordering, it implies that the minimum element must be one of the roots.

- Find-Min(H): look through all the roots in $O(\log n)$ time. (It turns out we can always maintain a pointer to the tree with smallest root to make this in $O(1)$ time, and updating this pointer when necessary doesn't increase any of the other bounds. But that's not the interesting thing about binomial heaps.)
- Meld(H_1, H_2): let's use the correspondence to binary counters. First, a warmup: what is H_1 and H_2 are both simply order k binomial trees? Then we know that their union has $2^k + 2^k = 2^{k+1}$ elements, so should be a single order $k+1$ binomial trees. And this is in fact really easy to achieve, since we can just choose whichever of the two roots is smaller, and add on the other B_k as its child. By definition this gives a binomial tree of order $k+1$, so we're done.

  Now let's look at the general case. We will use the correspondence with binary addition. We look at the orders from smallest to largest, starting with order 0. First, if neither of them has a B_0 then their sum doesn't. If exactly one of them has a B_0, then we use that same tree in the sum. If they both have a B_0, then we merge them into a B_1 and "carry" this into the next iteration.

So now consider iteration k. At this might point there might be $0, 1, 2$, or 3 order k binomial trees (depending on whether there was a carry tree and whether the starting heaps had order k binomial trees). If there are 0 then the union also has 0, and if there is 1 then we simply use that B_k as the B_k in the new heap. If there are 2, then we take their sum (in constant time) to create a new B_{k+1}, and we carry that to the next iteration and don't include any B_k in the new heap. If there are 3, then we choose one of them to be the B_k in the new heap (by convention we'll choose the carried-in tree), then take the sum of the other two to create a new B_{k+1} and carry this into the next iteration.

It's not hard to see that this gives a new binomial heap. Moreover, note that for each k, the running time was only constant. So the total time of Meld is just a constant times the number of trees, or $O(\log n)$. It turns out that it's also possible to prove an $O(1)$ amortized bound on Meld, but we won't do it here.

- Insert(H, x): create a new heap consisting of just x and do a Meld. So clearly $O(\log n)$ time, and just like with a binary counter the amortized cost of Insert is only $O(1)$.
- Extract-Min(H): The minimum element is one of the roots, so we can find it in $O(\log n)$ time. We can then delete it, which turns a single tree into a collection of binomial trees. But that's just another heap! So we can do a meld in time $O(\log n)$ to get back a single heap.
- Decrease-Key(H, x, k): just like binary heaps – change the key and swim up.
- Delete(H, x): Decrease the key of x to $-\infty$, then extract-min. Total time is $O(\log n)$.

8.4 Extensions

It turns out that there are even more advanced data structures, which can improve these running bounds. The most famous of these are *Fibonacci Heaps*, which gets Meld down to $O(1)$ and makes Insert $O(1)$ in the worst case. But Extract-Min becomes $O(\log n)$ only amortized, rather than worst case. In 2012 a new data structure known as *Strict Fibonacci Heaps* got rid of the amortization to give true worst case bounds. So in a strict Fibonacci Heap, all operations take $O(1)$ time except Extract-Min (which takes $O(\log n)$).