**600.363 Introduction to Algorithms / 600.463 Algorithms I** **Fall 2014**
**Homework #1** **Due:** September 9, 2014, 1:30pm

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Please start each problem on a new page, and include your name on each problem. You can submit on blackboard, under student assessment, or can bring your solutions to lecture.

Remember: you may work in groups of up to three people, but must write up your solution entirely on your own. Collaboration is limited to discussing the problems – you may not look at, compare, reuse, etc. any text from anyone else in the class. Please include your list of collaborators on the first page of your submission. You may use the internet to look up formulas, definitions, etc., but may not simply look up the answers online.

Please include proofs with all of your answers, unless stated otherwise.

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## 1 Asymptotic Notation (25 points)

For each of the following statements explain if it true or false and prove your answer. The base of log is 2 unless otherwise specified, and ln is $\log_e$.

(a) $5^{27}(n \log n + 24n^2) = \Theta(n^2)$

(b) $2^n = \Theta(e^{n+\sqrt{n}})$

(c) $5^n = \Theta(5^{(n+3)})$

(d) $\log(n^7) = \Theta(\ln(n^{10000000}))$

(e) $n \sin n = O(n)$

(f) Let $f, g$ be positive functions. Then $f(n) + g(n) = \Theta(\min(f(n), g(n)))$

(g) Let $f$ be a positive function. Then $f(n) + o(f(n)) = \Theta(f(n))$

## 2 Recurrences (25 pts)

Solve the following recurrences, giving your answer in $\Theta$ notation (so prove both an upper bound and a lower bound). For each of them you may assume $T(x) = 1$ for $x \leq 5$. Show your work.

(a) $T(n) = 3T(n/5) + n$

(b) $T(n) = 5T(n-3)$

(c) $T(n) = \sqrt{n}T(\sqrt{n}) + n$

(d) $T(n) = (\log n)T(n/\log n) + n$

(e) $T(n) = 2T(n/2) + n \log n$

## 3 Basic Proofs (25 pts)

(a) Let $A, B, C, D$ be sets. Prove that

$$(A \cap B) \cup (C \cap D) = (A \cup C) \cap (B \cup C) \cap (A \cup D) \cap (B \cup D)$$

(b) There are currently 61 students registered for the class. Prove that there are at least 6 students who have birthdays in the same month.

(c) I have a bucket with 50 balls, 15 of which are white and 35 of which are black. If I draw 8 balls at random from the bucket (all at one time), what is the probability that exactly three of them are white?

(d) Let $x_1, x_2, \dots x_n$ be real numbers. Prove that the maximum is at least the arithmetic mean, i.e. prove that

$$\max\{x_i\}_{i=1}^n \geq \frac{1}{n}\sum_{i=1}^n x_i.$$

## 4 Inversions (25 pts)

Given an array $[a_1, a_2, \dots, a_n]$, an inversion is a pair $(i, j)$ such that $i < j$ but $a_i > a_j$. For example, in the array $[5, 3, 2, 10]$ there are are three inversions $((1, 2), (1, 3), (2, 3))$. Note that if the array has no inversions if and only if it is sorted, so the number of inversions can be thought of as a measure of how well-sorted an array is.

(a) What is the expected number of an inversions in a random array? More formally, consider a random permutation of $n$ distinct elements $a_1, \dots, a_n$: what is the expected number of inversions? (Hint: let $X_{ij}$ be an indicator variable for $(i, j)$ being an inversion, and use linearity of expectations)

(b) Recall the insertion sort algorithm:

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for i = 1 to n
    j = i
    while j > 0 and A[j-1] > A[j]
        swap A[j] and A[j-1]
        j = j - 1
```

Suppose that our array has $d$ inversions. Give a lower bound on the running time of insertion sort in terms of $d$.

(c) Give an upper bound on the running time of insertion sort in terms of $n$ and $d$.

(d) What do parts (a), (b), and (c) imply about the average-case running time of insertion sort? (use $\Theta$ notation).