Trade-off between multiple-copy transformation and entanglement catalysis

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We demonstrate that multiple copies of a bipartite entangled pure state may serve as a catalyst for certain entanglement transformations while a single copy cannot. Such a state is termed a "multiple-copy catalyst" for the transformations. A trade-off between the number of copies of source state and that of the catalyst is also observed. These results can be generalized to probabilistic entanglement transformations directly.

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I. INTRODUCTION

In recent years, more and more applications of quantum information processing, such as quantum cryptography [1], quantum superdense coding [2], and quantum teleportation [3], have led us to view quantum entanglement as a new kind of physical resource [4]. One of the central problems about quantum entanglement is to find the conditions under which an entangled state could be converted into another one by using local quantum operations and classical communication (LOCC) only. Bennett and his collaborators [5–7] made significant progress in attacking this challenging problem for the asymptotic setting, while for the deterministic transformations, the first important step was made by Nielsen in Ref. [8], where he found a necessary and sufficient condition for a bipartite entangled pure state to be transformed to another pure one deterministically, under the constraint of LOCC. More precisely, suppose that Alice and Bob share an entangled state $|\psi\rangle$, and they want to transform it into another state $|\varphi\rangle$ by using only local quantum operations on their own subsystems and classical communication between them. Nielsen proved that the two parties can finish this task successfully—i.e., transforming $|\psi\rangle$ to $|\varphi\rangle$ with certainty under LOCC—if and only if $\psi \prec \varphi$, where ψ and φ denote the Schmidt coefficient vectors of $|\psi\rangle$ and $|\varphi\rangle$, respectively. Here the symbol "$\prec$" stands for "majorization relation," which is a vast topic in linear algebra. For details about majorization, see Refs. [9,10].

Nielsen's result implies that there can be two entangled pure states—say, $|\psi\rangle$ and $|\varphi\rangle$—such that they are incomparable in the sense that neither the transformation of $|\psi\rangle$ to $|\varphi\rangle$ nor the transformation of $|\varphi\rangle$ to $|\psi\rangle$ can be realized with certainty. For transformations between incomparable states, Vidal [11] generalized Nielsen's result to a probabilistic version and established an explicit expression to calculate the maximal conversion probability.

Shortly after Nielsen's work, a startling phenomenon of entanglement—namely, entanglement catalysis, or ELOCC—was discovered by Jonathan and Plenio [12]. This phenomenon can be understood as follows. Let $|\psi\rangle$ and $|\varphi\rangle$ be two bipartite entangled pure states such that the transformation of $|\psi\rangle$ to $|\varphi\rangle$ cannot be realized with certainty under LOCC. Then Jonathan and Plenio demonstrated that sometimes one may use an auxiliary entangled state $|\phi\rangle$, known as a catalyst, to make the above transformation possible without being consuming at all. In fact, the transformation in the presence of $|\phi\rangle$ is of the form $|\psi\rangle\otimes|\phi\rangle \to |\varphi\rangle\otimes|\phi\rangle$, from which one can easily see that the catalyst state $|\phi\rangle$ is not modified during the process. A concrete example is as follows. Take $|\psi\rangle = \sqrt{0.4}|00\rangle + \sqrt{0.4}|11\rangle + \sqrt{0.1}|22\rangle + \sqrt{0.1}|33\rangle$ and $|\varphi\rangle = \sqrt{0.5}|00\rangle + \sqrt{0.25}|11\rangle + \sqrt{0.25}|22\rangle$. We know that $|\psi\rangle$ cannot be transformed to $|\varphi\rangle$ with certainty under LOCC, but if another entangled state $|\phi\rangle = \sqrt{0.6}|44\rangle + \sqrt{0.4}|55\rangle$ is introduced, then the transformation of $|\psi\rangle\otimes|\phi\rangle$ to $|\varphi\rangle\otimes|\phi\rangle$ can be realized with certainty because $\psi\otimes\phi \prec \varphi\otimes\phi$. The role of the state $|\phi\rangle$ in this transformation is similar to a catalyst in a chemical process since it can help the entanglement transformation process without being consumed. In the same paper, Jonathan and Plenio also showed that the use of a catalyst can improve the maximal conversion probability when the transformation cannot be realized with certainty even with the help of a catalyst. The mathematical structure of entanglement catalysis was thoroughly studied in Ref. [13].

Bandyopadhyay *et al.* found another interesting phenomenon [14]: sometimes multiple copies of the source state may be transformed into the same number of copies of the target state although the transformation cannot happen for a single copy. Such a phenomenon is called "nonasymptotic bipartite pure-state entanglement transformation" in Ref. [14]. More intuitively, this phenomenon can also be called "multiple-copy entanglement transformation," or MLOCC for short. Take the above states $|\psi\rangle$ and $|\varphi\rangle$ as an example. It is not difficult to check that the transformation of $|\psi\rangle^{\otimes 3}$ to $|\varphi\rangle^{\otimes 3}$ occurs with certainty by Nielsen's theorem. That is, when Alice and Bob prepare three copies of $|\psi\rangle$ instead of just a single one, they can transform these three copies all together into the same number of copies of $|\varphi\rangle$ by LOCC. This simple example means that the effect of catalyst can, at least in the above situation, be implemented by preparing a sufficiently large number of copies of the original state and transforming these copies together. Some important aspects of MLOCC were investigated in Ref. [14].

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In this paper we examine the catalysis power when multiple copies of a catalyst state are available. What was discovered by Bandyopadhyay *et al.* is that sometimes the effect of catalysis can be implemented by increasing the number of copies of the source state, whereas we present some examples to show another interesting phenomenon: a large enough number of copies of an entangled pure state may act as a catalyst although a single copy cannot. Such an entangled pure state can be called a *multiple-copy catalyst*. More formally, if $|\phi\rangle$ is not a catalyst for the transformation of $|\psi\rangle$ to $|\varphi\rangle$, but there is an integer $m>1$ such that $|\phi\rangle^{\otimes m}$ is a catalyst for the same transformation, then $|\phi\rangle$ is called a multiple-copy catalyst for the transformation of $|\psi\rangle$ to $|\varphi\rangle$. A necessary condition for a given entangled pure state to be a multiple-copy catalyst for a specific transformation is obtained.

It is worth noting that both ways of enabling entanglement transformations in Ref. [14] and in the present paper are increasing the number of copies of states. The essential difference is that in Ref. [14] the number of copies of the source state is increased while in this paper we consider increasing the number of catalysts. A lot of heuristic examples lead us to find a trade-off between the number of copies of the original entangled state and that of the catalyst. As is expected, the more original-state copies are provided, the fewer catalyst copies are needed and vice versa.

A similar phenomenon also exists in the case of probabilistic entanglement transformations. We show by examples that sometimes the combination of MLOCC and ELOCC can increase the maximal conversion probability efficiently. We also present a necessary condition for when the combination of multiple-copy transformations and entanglement-assisted transformations has advantages over pure LOCC transformations.

The rest of the paper is organized as follows. In Sec. II, we study the combination of MLOCC and ELOCC in deterministic transformations. These results are generalized to probabilistic ones in Sec. III. The paper is concluded in Sec. IV with some open problems that may be of interest for further study.

II. COMBINING MLOCC WITH ELOCC: DETERMINISTIC CASE

The main purpose of this section is to demonstrate the effect of multiple-copy catalysis. We accomplish this goal by giving an explicit example, which confirms the existence of a multiple-copy catalyst. Then we further combine multiple-copy transformation and entanglement catalysis together and show that a trade-off exists between the number of copies of a multiple-copy catalyst and that of the source state in the entanglement transformation. A necessary condition for an auxiliary state being a multiple-copy catalyst for the given transformation is also presented.

For the sake of convenience, we present here Nielsen's theorem [8] as a lemma since it will be used frequently to analyze the possibility of entanglement transformations latter.

Lemma 1. Let $|\psi\rangle=\sum_{i=1}^{n}\sqrt{\alpha_i}|i\rangle|i\rangle$ and $|\varphi\rangle=\sum_{i=1}^{n}\sqrt{\beta_i}|i\rangle|i\rangle$ be pure bipartite states with the Schmidt coefficient vectors $\psi=(\alpha_1,\ldots,\alpha_n)$ and $\varphi=(\beta_1,\ldots,\beta_n)$, where $\alpha_1\geqslant\cdots\geqslant\alpha_n\geqslant 0$ and $\beta_1\geqslant\cdots\geqslant\beta_n\geqslant 0$. Then there exists a transformation that converts $|\psi\rangle$ into $|\varphi\rangle$ with certainty under LOCC if and only if $\psi\prec\varphi$—i.e.,

$$\sum_{i=1}^{l}\alpha_i\leqslant\sum_{i=1}^{l}\beta_i,\quad 1\leqslant l\leqslant n, \tag{1}$$

with equality when $l=n$.

Nielsen's theorem establishes a connection between the transformation of $|\psi\rangle$ to $|\varphi\rangle$ and the mathematical relation $\psi\prec\varphi$. Intuitively, we often write $|\psi\rangle\prec|\varphi\rangle$ instead of $\psi\prec\varphi$. From that one can immediately deduce that the transformation of $|\psi\rangle$ to $|\varphi\rangle$ can be achieved with certainty under LOCC.

As a useful application of Nielsen's theorem, we present a technical lemma as follows.

Lemma 2. Let $|\psi\rangle$ and $|\varphi\rangle$ be two bipartite entangled pure states. If $|\psi\rangle^{\otimes p}\prec|\varphi\rangle^{\otimes p}$ for each $p=k,k+1,\ldots,2k-1$, then $|\psi\rangle^{\otimes p}\prec|\varphi\rangle^{\otimes p}$ for all $p\geqslant k$.

In other words, to check whether $|\psi\rangle^{\otimes p}\prec|\varphi\rangle^{\otimes p}$ holds for every $p\geqslant k$, one only needs to check k values of p—i.e., $p=k,\ldots,2k-1$.

Proof. By Nielsen's theorem and the assumptions, to prove that $|\psi\rangle^{\otimes p}\prec|\varphi\rangle^{\otimes p}$ for every $p\geqslant k$, we only need to show that the transformation of $|\psi\rangle^{\otimes p}$ to $|\varphi\rangle^{\otimes p}$ can be realized with certainty for any $p\geqslant 2k$. For this purpose, we uniquely decompose the positive integer p such that $p\geqslant 2k$ as

$$p=(r-1)k+(k+s),\quad r\geqslant 2\quad\text{and}\quad 0\leqslant s\leqslant k-1. \tag{2}$$

Now an explicit protocol implementing the transformation of $|\psi\rangle^{\otimes p}$ to $|\varphi\rangle^{\otimes p}$ with certainty under LOCC consists of the following two steps.

(i) Perform $(r-1)$ times of the transformation of $|\psi\rangle^{\otimes k}$ to $|\varphi\rangle^{\otimes k}$.

(ii) Perform one time of the transformation of $|\psi\rangle^{\otimes k+s}$ to $|\varphi\rangle^{\otimes k+s}$.

By Nielsen's theorem and the assumptions again, we know that both the transformations in (i) and (ii) can be realized with certainty by LOCC. That completes the proof of lemma 2. □

It is worth noting that the conditions in lemma 2 are also necessary in general. In fact, as pointed out by Leung and Smolin in Ref. [15], the majorization relation is not monotonic in general in the sense that $|\psi\rangle^{\otimes k}\prec|\varphi\rangle^{\otimes k}$ does not always imply $|\psi\rangle^{\otimes k+1}\prec|\varphi\rangle^{\otimes k+1}$. Thus, to guarantee that $|\psi\rangle^{\otimes p}\prec|\varphi\rangle^{\otimes p}$ holds for every $p\geqslant k$, one needs to check all k conditions.

Now we begin to examine the catalysis power when multiple copies of the catalyst state are available. In particular, the following example indicates the existence of a multiple-copy catalyst.

Example 1. Suppose that the original entangled state owned by Alice and Bob is

$$|\psi\rangle=\sqrt{0.4}|00\rangle+\sqrt{0.4}|11\rangle+\sqrt{0.1}|22\rangle+\sqrt{0.1}|33\rangle, \tag{3}$$

and the final state they want to transform $|\psi\rangle$ into is

$$|\varphi\rangle = \sqrt{0.5}|00\rangle + \sqrt{0.25}|11\rangle + \sqrt{0.22}|22\rangle + \sqrt{0.03}|33\rangle. \quad (4)$$

This example is very close to the original one used by Jonathan and Plenio [12] to demonstrate the effect of catalysis. One may think that Alice and Bob could realize the transformation of $|\psi\rangle$ to $|\varphi\rangle$ with a 2×2 catalyst, as in the original example in Ref. [12]. Unfortunately, it is not the case since the small deviation violates the condition of the existence of a 2×2 catalyst [16]. However, we can find a 3×3 state

$$|\phi_1\rangle = \sqrt{\frac{50}{103}}|44\rangle + \sqrt{\frac{30}{103}}|55\rangle + \sqrt{\frac{23}{103}}|66\rangle \quad (5)$$

such that $|\psi\rangle\otimes|\phi_1\rangle \prec |\varphi\rangle\otimes|\phi_1\rangle$.

Moreover, by a routine calculation, we may observe that

$$|\psi\rangle^{\otimes k} \nprec |\varphi\rangle^{\otimes k}, \quad 1 \leqslant k \leqslant 4, \quad (6)$$

but

$$|\psi\rangle^{\otimes k} \prec |\varphi\rangle^{\otimes k}, \quad 5 \leqslant k \leqslant 9 \quad (7)$$

holds. Thus Eq. (7) is true for any $k \geqslant 5$ by lemma 2. Again, this shows that the effect of a catalyst can be implemented by increasing the number of copies of the source state in a transformation. We now further put

$$|\phi_2\rangle = \sqrt{0.6}|44\rangle + \sqrt{0.4}|55\rangle, \quad (8)$$

which is certainly not a catalyst for the transformation mentioned above. An interesting thing here is that $|\phi_2\rangle^{\otimes 5}$ does serve as a catalyst for the transformation of $|\psi\rangle$ to $|\varphi\rangle$ because an easy calculation shows that $|\psi\rangle\otimes|\phi_2\rangle^{\otimes 5} \prec |\varphi\rangle \otimes|\phi_2\rangle^{\otimes 5}$. Of course, $|\phi_2\rangle^{\otimes 5}$ is not the optimal one in the sense that its dimension is not the minimum among all catalysts. This phenomenon indicates that increasing the number of an entangled pure state may strictly broaden the power of its catalysis. □

In the next example, we combine MLOCC with ELOCC, and show that a trade-off exists between the number of copies of source state and that of the catalyst.

Example 2. Suppose that Alice and Bob share some copies of source state $|\psi\rangle$ as in Eq. (3), and they want to transform it to the same number of copies of

$$|\varphi\rangle = \sqrt{0.5}|00\rangle + \sqrt{0.25}|11\rangle + \sqrt{0.2}|22\rangle + \sqrt{0.05}|33\rangle \quad (9)$$

by LOCC. Suppose that the only states they can borrow from a catalyst banker are some copies of $|\phi_2\rangle$ in Eq. (8). Can Alice and Bob realize their task? Notice that

$$|\psi\rangle^{\otimes 5} \nprec |\varphi\rangle^{\otimes 5} \quad \text{but } |\psi\rangle^{\otimes k} \prec |\varphi\rangle^{\otimes k}, \quad 6 \leqslant k \leqslant 11. \quad (10)$$

Applying lemma 2 yields that if the number of available copies of $|\psi\rangle$ is larger than or equal to 6, then Alice and Bob always can realize their task by themselves without borrowing any catalyst. But if they only own 5 copies of $|\psi\rangle$, they cannot realize the transformation even if joint operations on the 5 copies are performed. It is easy to check that borrowing one copy of $|\phi_2\rangle$ is enough for Alice and Bob's task because $|\psi\rangle^{\otimes 5}\otimes|\phi_2\rangle \prec |\varphi\rangle^{\otimes 5}\otimes|\phi_2\rangle$. Similarly, when they only own 4 copies of $|\psi\rangle$, it is sufficient to finish the task successfully by borrowing 2 copies of $|\phi_2\rangle$. For the case that 3 copies of $|\psi\rangle$ are owned by Alice and Bob, it is easy to see that 3 copies of $|\phi_2\rangle$ are not enough for their purpose and the minimal number of $|\phi_2\rangle$ is 4. Finally, when Alice and Bob own only one copy of $|\psi\rangle$, using 6–10 copies of $|\phi_2\rangle$ cannot achieve the task. We conclude that they must borrow at leat 11 copies of $|\phi_2\rangle$ from the catalyst banker since the relation $|\psi\rangle\otimes|\phi_2\rangle^{\otimes k} \prec |\varphi\rangle\otimes|\phi_2\rangle^{\otimes k}$ holds only for $k \geqslant 11$. Here we have used Nielsen's theorem and the fact that if $|\phi\rangle^{\otimes k}$ is a catalyst for the transformation of $|\psi\rangle$ to $|\varphi\rangle$ then $|\phi\rangle^{\otimes p}$ is also a catalyst for the same transformation for any $p \geqslant k$. Alice and Bob must borrow a large number of catalysts to complete the transformation in this extreme case. This example illustrates a trade-off between the number of copies of original state and that of catalyst. □

The above two examples show that it will be very useful to know when a given entangled pure state can serve as a multiple-copy catalyst for a specific entanglement transformation. Unfortunately, such a characterization is not known at present. Nevertheless, we can give a necessary condition for the existence of a multiple-copy catalyst.

Before presenting this necessary condition, we introduce some useful notations. We define $x^{\downarrow}$ as the vector which is obtained by rearranging the components of x into nonincreasing order. A useful fact about this notation is that $x^{\downarrow} = y^{\downarrow}$ if and only if the components of x are exactly the same as those of y. In other words, they are equivalent up to a permutation. For any bipartite entangled pure states $|\psi\rangle$ and $|\varphi\rangle$ with the ordered Schmidt coefficient vectors $\psi^{\downarrow} = (\alpha_1, \ldots, \alpha_n)$ and $\varphi^{\downarrow} = (\beta_1, \ldots, \beta_n)$, we define a set of indices as

$$L_{\psi,\varphi} = \left\{ l{:}1 \leqslant l < n \quad \text{and} \quad \sum_{j=1}^{l} \alpha_j > \sum_{j=1}^{l} \beta_j \right\}. \quad (11)$$

Intuitively, for any $l \in L_{\psi,\varphi}$, the sum of the l largest components of ψ is strictly larger than that of φ. So $|\psi\rangle$ and $|\varphi\rangle$ are incomparable if and only if $L_{\psi,\varphi} \neq \emptyset$ and $L_{\varphi,\psi} \neq \emptyset$.

The following lemma is interesting in its own right. It gives us a necessary condition for a bipartite entangled pure state $|\phi\rangle$ with Schmidt coefficients $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_k > 0$ to be a catalyst for a given transformation.

Lemma 3. Let $|\psi\rangle$ and $|\varphi\rangle$ be two incomparable states. If $|\phi\rangle$ is a catalyst for the transformation of $|\psi\rangle$ to $|\varphi\rangle$, then for any $l \in L_{\psi,\varphi}$, it holds that $\gamma_1/\gamma_k > \beta_l/\beta_{l+1}$ and

$$\frac{\gamma_1}{\gamma_i} > \frac{\beta_l}{\beta_{l+1}} \quad \text{or} \quad \frac{\gamma_i}{\gamma_{i+1}} < \frac{\beta_1}{\beta_l} \quad (12)$$

and

$$\frac{\gamma_{i+1}}{\gamma_k} > \frac{\beta_l}{\beta_{l+1}} \quad \text{or} \quad \frac{\gamma_i}{\gamma_{i+1}} < \frac{\beta_{l+1}}{\beta_n} \quad (13)$$

for $i = 1, \ldots, k-1$.

Proof. By contradiction, suppose that one of the following holds.

Case (a): there exist $l_0 \in L_{\psi,\varphi}$ and $1 \leqslant i_0 \leqslant k-1$ such that either Eq. (12) or Eq. (13) does not hold.

Case (b): there exists $l_0 \in L_{\psi,\varphi}$ such that $\gamma_1/\gamma_k \leqslant \beta_{l_0}/\beta_{l_0+1}$.

We only need to prove that both cases (a) and (b) contradict the assumption $|\psi\rangle\otimes|\phi\rangle\prec|\varphi\rangle\otimes\phi\rangle$.

First, we deal with case (a). Let us decompose ψ into two shorter vectors ψ' and ψ''—that is, $\psi=(\psi',\psi'')$—such that $\psi'=(\alpha_1,\ldots,\alpha_{l_0})$ and $\psi''=(\alpha_{l_0+1},\ldots,\alpha_n)$. φ is similarly decomposed as $\varphi=(\varphi',\varphi'')$. We also decompose $\phi=(\phi',\phi'')$, where $\phi'=(\gamma_1,\ldots,\gamma_{i_0})$ and $\phi''=(\gamma_{i_0+1},\ldots,\gamma_k)$.

Since $\varphi\otimes\phi=(\varphi',\varphi'')\otimes(\phi',\phi'')$, one can easily check that the components of $\varphi\otimes\phi$ are exactly the same as those of $(\varphi'\otimes\phi',\varphi'\otimes\phi'',\varphi''\otimes\phi',\varphi''\otimes\phi'')$ by a simple algebraic calculation. By our notations introduced above, we always have

$$(\varphi\otimes\phi)^{\downarrow}=(\varphi'\otimes\phi',\varphi'\otimes\phi'',\varphi''\otimes\phi',\varphi''\otimes\phi'')^{\downarrow}. \tag{14}$$

Notice that the minimal component of $\varphi'\otimes\phi'$ is $\beta_{l_0}\gamma_{i_0}$, while the maximal components of $\varphi'\otimes\phi''$, $\varphi''\otimes\phi'$, and $\varphi''\otimes\psi''$ are $\beta_1\gamma_{i_0+1}$, $\beta_{l_0+1}\gamma_1$, and $\beta_{l_0+1}\gamma_{i_0+1}$, respectively.

To finish the proof of case (a), it suffices to consider the following two subcases.

Subcase (a.1): Equation (12) is not satisfied, that is,

$$\gamma_1/\gamma_{i_0}\leqslant\beta_{l_0}/\beta_{l_0+1}\quad\text{and}\quad\gamma_{i_0}/\gamma_{i_0+1}\geqslant\beta_1/\beta_{l_0}, \tag{15}$$

then

$$\beta_{l_0}\gamma_{i_0}\geqslant\max\{\beta_1\gamma_{i_0+1},\beta_{l_0+1}\gamma_1,\beta_{l_0+1}\gamma_{i_0+1}\}, \tag{16}$$

which implies that the minimal component of $\varphi'\otimes\phi'$ is not less than the maximal components of $\varphi'\otimes\phi''$, $\varphi''\otimes\phi'$ and $\varphi''\otimes\phi''$. By Eqs. (14) and (16), the largest i_0l_0 components of $\varphi\otimes\phi$ are just the components of $\varphi'\otimes\phi'$. So

$$\begin{aligned}\sum_{j=1}^{i_0l_0}(\varphi\otimes\phi)_j^{\downarrow}&=\sum_{j=1}^{i_0l_0}(\varphi'\otimes\phi')_j^{\downarrow}\\&=\left(\sum_{j=1}^{l_0}\beta_j\right)\left(\sum_{j=1}^{i_0}\gamma_j\right)<\left(\sum_{j=1}^{l_0}\alpha_j\right)\left(\sum_{j=1}^{i_0}\gamma_j\right)\\&=\sum_{j=1}^{i_0l_0}(\psi'\otimes\phi')_j^{\downarrow}\leqslant\sum_{j=1}^{i_0l_0}(\psi\otimes\phi)_j^{\downarrow},\end{aligned} \tag{17}$$

where the strict inequality follows from $l_0\in L_{\psi,\varphi}$, while the last inequality is by the definition of $\sum_{j=1}^{i_0l_0}(\psi\otimes\phi)_j^{\downarrow}$. It follows that $|\psi\rangle\otimes|\phi\rangle\not\prec|\varphi\rangle\otimes|\phi\rangle$, a contradiction.

Subcase (a.2): Equation (13) is not satisfied. Then by similar arguments we can verify that the least $(k-i_0)(n-l_0)$ components of $\varphi\otimes\phi$ are just the components of $\varphi''\otimes\phi''$, and thus $|\psi\rangle\otimes|\phi\rangle\not\prec|\varphi\rangle\otimes|\phi\rangle$, by considering the sum of the least $(k-i_0)(n-l_0)$ components of $\varphi\otimes\phi$. This is also a contradiction.

Now we deal with case (b). In this case, $\phi'=\phi$ and ϕ'' disappears. With almost the same arguments as in case (a.1), we have that $|\psi\rangle\otimes|\phi\rangle\not\prec|\varphi\rangle\otimes|\phi\rangle$, again a contradiction. That completes the proof of lemma 3. □

In the above lemma, if we take $i=1$, then from Eq. (12) we have $\gamma_1/\gamma_2<\beta_1/\beta_l$. Similarly, taking $i=k-1$ leads us to $\gamma_{k-1}/\gamma_k<\beta_{l+1}/\beta_n$ from Eq. (13). Consequently, we have the following corollary.

Corollary 1. Let $|\psi\rangle$ and $|\varphi\rangle$ be two incomparable states. If $|\phi\rangle$ is a catalyst for the transformation of $|\psi\rangle$ to $|\varphi\rangle$, then, for any $l\in L_{\psi,\varphi}$,

$$\frac{\gamma_1}{\gamma_2}<\frac{\beta_1}{\beta_l}\quad\text{and}\quad\frac{\gamma_{k-1}}{\gamma_k}<\frac{\beta_{l+1}}{\beta_n}. \tag{18}$$

The following theorem indicates that the condition in Eq. (18) is also necessary for $|\phi\rangle$ to be a multiple-copy catalyst for the transformation of $|\psi\rangle$ to $|\varphi\rangle$.

Theorem 1. Let $|\psi\rangle$ and $|\varphi\rangle$ be two incomparable states. If $|\phi\rangle$ is a multiple-copy catalyst for the transformation of $|\psi\rangle$ to $|\varphi\rangle$, then for any $l\in L_{\psi,\varphi}$, Eq. (18) holds.

Proof. If $|\phi\rangle$ is a multiple-copy catalyst for the transformation of $|\psi\rangle$ to $|\varphi\rangle$, then there exists a positive integer m such that $|\phi\rangle^{\otimes m}$ is a catalyst for the same transformation. By corollary 1, it follows that

$$\frac{(\phi^{\otimes m})_1^{\downarrow}}{(\phi^{\otimes m})_2^{\downarrow}}<\frac{\beta_1}{\beta_l} \tag{19}$$

and

$$\frac{(\phi^{\otimes m})_{k^m-1}^{\downarrow}}{(\phi^{\otimes m})_{k^m}^{\downarrow}}<\frac{\beta_{l+1}}{\beta_n} \tag{20}$$

for any $l\in L_{\psi,\varphi}$.

It is easy to check that

$$\frac{(\phi^{\otimes m})_1^{\downarrow}}{(\phi^{\otimes m})_2^{\downarrow}}=\frac{\gamma_1^m}{\gamma_2\gamma_1^{m-1}}=\frac{\gamma_1}{\gamma_2} \tag{21}$$

and

$$\frac{(\phi^{\otimes m})_{k^m-1}^{\downarrow}}{(\phi^{\otimes m})_{k^m}^{\downarrow}}=\frac{\gamma_k^{m-1}\gamma_{k-1}}{\gamma_k^m}=\frac{\gamma_{k-1}}{\gamma_k}. \tag{22}$$

Combining Eqs. (19)–(22), we have the validity of Eq. (18). This completes the proof of theorem 1. □

With the help of theorem 1, we are able to find a state $|\phi\rangle$ such that it is a multiple-copy catalyst for the transformation of $|\psi\rangle^{\otimes k}$ to $|\varphi\rangle^{\otimes k}$ with some $k>1$, but not for the transformation of $|\psi\rangle$ to $|\varphi\rangle$. Intuitively, a multiple-copy transformation can be catalyzed more easily than a single-copy transformation.

Example 3. Take the source state as

$$|\psi'\rangle=\frac{1}{\sqrt{1.01}}(|\psi\rangle+\sqrt{0.01}|44\rangle), \tag{23}$$

while the target as

$$|\varphi'\rangle=\frac{1}{\sqrt{1.01}}(|\varphi\rangle+\sqrt{0.01}|44\rangle), \tag{24}$$

where $|\psi\rangle$ and $|\varphi\rangle$ are defined as Eq. (3) and Eq. (9), respectively. We choose

$$|\phi_3\rangle=\sqrt{0.7}|55\rangle+\sqrt{0.3}|66\rangle. \tag{25}$$

A simple calculation shows that $|\phi_3\rangle$ is a catalyst for a 5-copy transformation (i.e., the transformation of $|\psi'\rangle^{\otimes 5}$ to

$|\varphi'\rangle^{\otimes 5}$), and $|\phi_3\rangle^{\otimes 2}$ is a catalyst both for a 4-copy transformation and for a 3-copy transformation. It is obvious that $L_{\psi',\varphi'}=\{2\}$, $\varphi'=1/1.01(0.5,0.25,0.2,0.05,0.01)$ and $\phi_3=(0.7,0.3)$. So

$$\frac{\gamma_1}{\gamma_2}=\frac{0.7}{0.3}>\frac{0.5}{0.25}=\frac{\beta_1}{a_2},$$

which yields that the condition in Eq. (18) is violated. Thus, by theorem 1, it follows that $|\phi_3\rangle$ is not a multiple-copy catalyst for the transformation of $|\psi'\rangle$ to $|\varphi'\rangle$. In other words, for arbitrarily large q, the transformation of $|\psi'\rangle\otimes|\phi_3\rangle^{\otimes q}$ to $|\varphi'\rangle\otimes|\phi_3\rangle^{\otimes q}$ cannot be achieved with certainty. □

III. COMBINING MLOCC WITH ELOCC: PROBABILISTIC CASE

We considered deterministic transformations in the last section. In this section, let us turn to examine entanglement transformations with probability strictly less than 1.

Recall Vidal's theorem from Ref. [11] that the maximal conversion probability of transforming $|\psi\rangle$ to $|\varphi\rangle$ under LOCC is given by

$$P_{max}(|\psi\rangle\to|\varphi\rangle)=\min_{1\leq l\leq n}\frac{E_l(|\psi\rangle)}{E_l(|\varphi\rangle)}, \tag{26}$$

where $E_l(|\psi\rangle)=\Sigma_{i=l}^{n}\alpha_i$ and $\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_n$ are the Schmidt coefficients of $|\psi\rangle$.

Let $\lambda\in(0,1)$. We call $|\phi\rangle$ a λ *catalyst* for the transformation of $|\psi\rangle$ to $|\varphi\rangle$ if

$$P_{max}(|\psi\rangle\otimes|\phi\rangle\to|\varphi\rangle\otimes|\phi\rangle)\geq\lambda. \tag{27}$$

Furthermore, if $|\phi\rangle^{\otimes k}$ serves as a λ catayst for some integer $k>1$, then we say that $|\phi\rangle$ is a *multiple-copy* λ *catalyst* for this transformation.

We say that a transformation $|\psi\rangle$ of $|\varphi\rangle$ can attain probability λ by MLOCC if there exists a positive integer k such that

$$P_{max}(|\psi\rangle^{\otimes k}\to|\varphi\rangle^{\otimes k})\geq\lambda^k. \tag{28}$$

Notice that if the maximal conversion probability from $|\psi\rangle$ to $|\varphi\rangle$ by LOCC is λ, then the right-hand side of the above equation is just the maximal conversion probability of transforming $|\psi\rangle^{\otimes k}$ into $|\varphi\rangle^{\otimes k}$ separately—that is, in a way where no collective operations on the k copies are performed. Thus the intuition behind the above definition is that with the help of MLOCC, the average probability of a single-copy transformation is not less than λ.

With the above preliminaries, the results obtained in Sec. II can be directly extended to the probabilistic case. The following example, first considered by Jonathan and Plenio in [12], demonstrates the existence of multiple-copy λ catalysts. It also shows that the presence of a multiple-copy λ catalyst and multiple copies of the source state can increase the maximal conversion probability efficiently.

Example 4. Let $|\psi\rangle=\sqrt{0.6}|00\rangle+\sqrt{0.2}|11\rangle+\sqrt{0.2}|22\rangle$ and $|\varphi\rangle=\sqrt{0.5}|00\rangle+\sqrt{0.4}|11\rangle+\sqrt{0.1}|22\rangle$. By Vidal's theorem, we have that $P_{max}(|\psi\rangle\to|\varphi\rangle)=0.80$. However, with the aid of an entangled state $|\phi\rangle=\sqrt{0.65}|33\rangle+\sqrt{0.35}|44\rangle$, the maximal conversion probability becomes $P_{max}(|\psi\rangle\otimes|\phi\rangle\to|\varphi\rangle\otimes|\phi\rangle)=0.904$, which means that $|\phi\rangle$ is a 0.904 catalyst for the transformation of $|\psi\rangle$ to $|\varphi\rangle$. Can Alice and Bob increase their conversion probability to 0.985? A careful analysis shows that the transformation of $|\psi\rangle$ to $|\varphi\rangle$ does not have any 2×2 0.985 catalyst [17]. Fortunately, $|\phi\rangle$ is a multiple-copy 0.985 catalyst since

$$P_{max}(|\psi\rangle\otimes|\phi\rangle^{\otimes 19}\to|\varphi\rangle\otimes|\phi\rangle^{\otimes 19})\geq 0.985. \tag{29}$$

Suppose now that Alice and Bob share two copies of $|\psi\rangle$. According to our definition, the transformation of $|\psi\rangle$ to $|\varphi\rangle$ can attain a probability $(0.8533)^{1/2}=0.9237$ under MLOCC since

$$P_{max}(|\psi\rangle^{\otimes 2}\to|\varphi\rangle^{\otimes 2})=0.8533. \tag{30}$$

If we combine a catalyst-assisted transformation and multiple-copy one together, the maximal conversion probability can increase efficiently. For example,

$$P_{max}(|\psi\rangle^{\otimes 2}\otimes|\phi\rangle^{\otimes 3}\to|\varphi\rangle^{\otimes 2}\otimes|\phi\rangle^{\otimes 3})=0.9535. \tag{31}$$

This implies that the transformation of $|\psi\rangle$ to $|\varphi\rangle$ can attain the probability $0.9535^{1/2}=0.9765$ under the combination of MLOCC and ELOCC. In contrast to that, a pure MLOCC needs at least 7 copies of $|\psi\rangle$ to attain the probability 0.985. □

Next, let us turn to another interesting question: is it always useful to combine a catalyst-assisted transformation with a multiple-copy transformation? The above two examples give some hints at a positive answer to the question. However, the next theorem indicates that such an improvement does not always happen. This theorem is a generalization of lemma 4 in [12] which says that the presence of catalysts cannot always increase the conversion probability. We should point out that a similar result has also been obtained in [14].

For any bipartite entangled pure states $|\psi\rangle$ and $|\varphi\rangle$, we define

$$P_{max}^{E}(|\psi\rangle\to|\varphi\rangle)=\sup_{|\phi\rangle}P_{max}(|\psi\rangle\otimes|\phi\rangle\to|\varphi\rangle\otimes|\phi\rangle). \tag{32}$$

Intuitively, $P_{max}^{E}(|\psi\rangle\to|\varphi\rangle)$ denotes the optimal conversion probability of transforming $|\psi\rangle$ to $|\varphi\rangle$ by using some catalyst.

Theorem 2. Let $|\psi\rangle$ and $|\varphi\rangle$ be two $n\times n$ states with the least Schmidt coefficients α_n and β_n, respectively. Then we have that

$$[P_{max}(|\psi\rangle\to|\varphi\rangle)]^p\leq P_{max}^{E}(|\psi\rangle^{\otimes p}\to|\varphi\rangle^{\otimes p})\leq\left(\frac{\alpha_n}{\beta_n}\right)^p \tag{33}$$

for any positive integer p.

Proof. The first inequality in Eq. (33) is obtained by performing the transformation of $|\psi\rangle^{\otimes p}$ to $|\varphi\rangle^{\otimes p}$ under LOCC separately. The second inequality in Eq. (33) can be proven as follows. Suppose that $|\phi\rangle$ is any entangled pure state with the least Schmidt coefficient $\gamma_k>0$. By Vidal's theorem, we obtain that

$$P_{max}(|\psi\rangle^{\otimes p} \otimes |\phi\rangle \rightarrow |\varphi\rangle^{\otimes p} \otimes |\phi\rangle) = \min_{1\leqslant l\leqslant n^p k} \frac{E_l(|\psi\rangle^{\otimes p} \otimes |\phi\rangle)}{E_l(|\varphi\rangle^{\otimes p} \otimes |\phi\rangle)} \leqslant \frac{E_{n^p k}(|\psi\rangle^{\otimes p} \otimes |\phi\rangle)}{E_{n^p k}(|\varphi\rangle^{\otimes p} \otimes |\phi\rangle)} = \frac{\alpha_n^p \gamma_k}{\beta_n^p \gamma_k} = \left(\frac{\alpha_n}{\beta_n}\right)^p, \tag{34}$$

where we have used the fact that $E_{n^p k}(|\psi\rangle^{\otimes p} \otimes |\phi\rangle) = \alpha_n^p \gamma_k$. The second inequality of Eq. (33) follows from Eqs. (32) and (34). This completes the proof of the theorem. □

Corollary 2. With the same assumption as in theorem 2, if $P_{max}(|\psi\rangle \rightarrow |\varphi\rangle) = \alpha_n / \beta_n$, then $P_{max}^E(|\psi\rangle^{\otimes p} \rightarrow |\varphi\rangle^{\otimes p}) = (\alpha_n / \beta_n)^p$.

In other words, even the combination of a multiple-copy transformation and catalyst-assisted transformation cannot increase the conversion probability. In fact, collective operations in this case have no advantages over individual operations.

An interesting application of corollary 2 is to deal with the case when $|\varphi\rangle$ is a maximally entangled state—that is, $|\varphi\rangle = (1/\sqrt{n})\Sigma_{i=1}^n |i\rangle|i\rangle$. The maximal conversion probability $P_{max}(|\psi\rangle \rightarrow |\varphi\rangle) = n\alpha_n$ cannot be increased by any combination of multiple-copy transformations and entanglement-assisted ones. Example 4 gives another application of the corollary. In fact, for any (3×3)-dimensional $|\psi\rangle$ and $|\varphi\rangle$, if $\alpha_3 < \beta_3$, then it follows from Vidal's theorem that $P_{max}(|\psi\rangle \rightarrow |\varphi\rangle) = \alpha_3 / \beta_3$. Hence, by the above corollary, $P_{max}^E(|\psi\rangle^{\otimes p} \rightarrow |\varphi\rangle^{\otimes p}) = (\alpha_3 / \beta_3)^p$, which is exponentially decreasing when p increases, as pointed out in Ref. [14].

IV. CONCLUSION

To summarize, we have demonstrated that in some cases multiple copies of an entangled state can serve as a catalyst although a single copy cannot. Such a state is called a multiple-copy catalyst. We have analyzed the power of combining MLOCC with ELOCC. Moreover, a trade-off between the number of copies of a source state and that of a catalyst is observed. We also show that the combination of MLOCC and ELOCC can increase the maximal conversion probability efficiently. Note that there are no analytical ways to find catalysts for a given transformation except for some special cases [16,18]. The notion of a multiple-copy catalyst sometimes may lead us to a possible way to seek an intended catalyst.

There are many open problems that may be of relevance. The most interesting one is, of course, what is the precise relation between MLOCC and ELOCC? Furthermore, is the combination of MLOCC and ELOCC always more powerful than MLOCC or ELOCC [18]? Another interesting one is to give a sufficient condition for a given entangled state to be a multiple-copy catalyst for a certain transformation.

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