1. (25 points) Define a two-dimensional Turing machine to be a TM where each of its tapes is an infinite grid (and the read/write head can move not only Left and Right but also Up and Down). Show that for every $T : \mathbb{N} \rightarrow \mathbb{N}$, any two-dimensional TM that runs in time $T(n)$ can be simulated by a standard (one-dimensional) TM in time $O(T(n)^2)$. Note: You may assume that the tapes of the two-dimensional TM start at $(0, 0)$ and can only access points with non-negative integer coordinates. The function $T(n)$ is not known in advance.

2. (25 points) Define a RAM Turing machine to be a Turing machine that has random access memory. We formalize this as follows: The machine has an infinite array $A$ that is initialized to all blanks. It accesses this array as follows: One of the machine’s work tapes is designated as the address tape. Also the machine has two special alphabet symbols denoted by $R$ and $W$ and an additional state we denote by $q_{access}$. Whenever the machine enters $q_{access}$, if its address tape contains $[i]R$ (where $[i]$ denotes the binary representation of $i$) then the value $A[i]$ is written in the cell next to the $R$ symbol. If its tape contains $[i]W\sigma$ (where $\sigma$ is some symbol in the machine’s alphabet) then $A[i]$ is set to the value $\sigma$.

Show that for every $T : \mathbb{N} \rightarrow \mathbb{N}$, any RAM TM that runs in time $T(n)$ can be simulated by a standard TM in time $O(T(n)^3)$.

3. (25 points) Let $T = \{ (M) : M$ is a TM that accepts $\alpha^R$ whenever it accepts $\alpha \}$, Show that $T$ is undecidable. Note: $\alpha^R$ is the reverse string of $\alpha$. You may assume the alphabet is $\{0, 1\}$.

4. (25 points) Define the language $C_{TM} = \{ (M_1, M_2) : M_1, M_2$ are two Turing machines such that $L(M_1) \cap L(M_2) = \emptyset \}$

Show that $C_{TM}$ is unrecognizable.