Wheeler graphs, part 3: Definition

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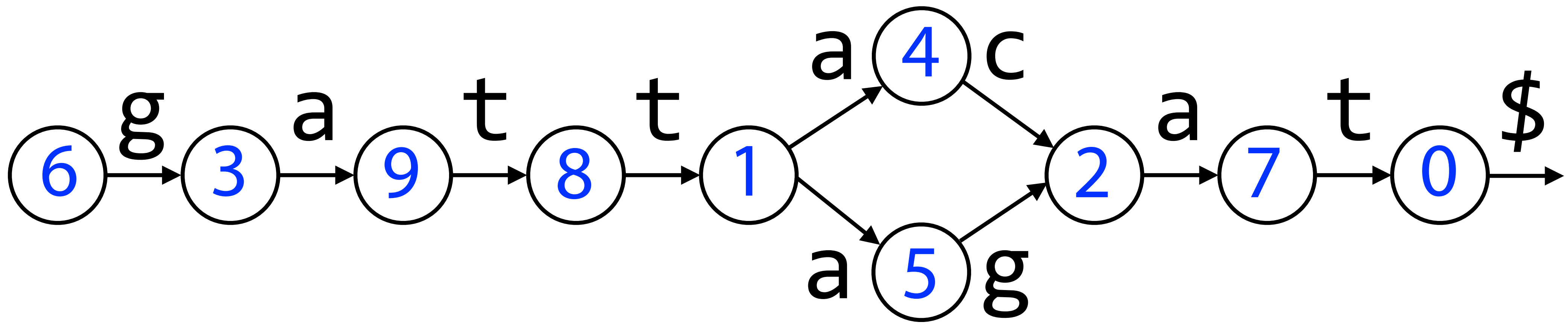
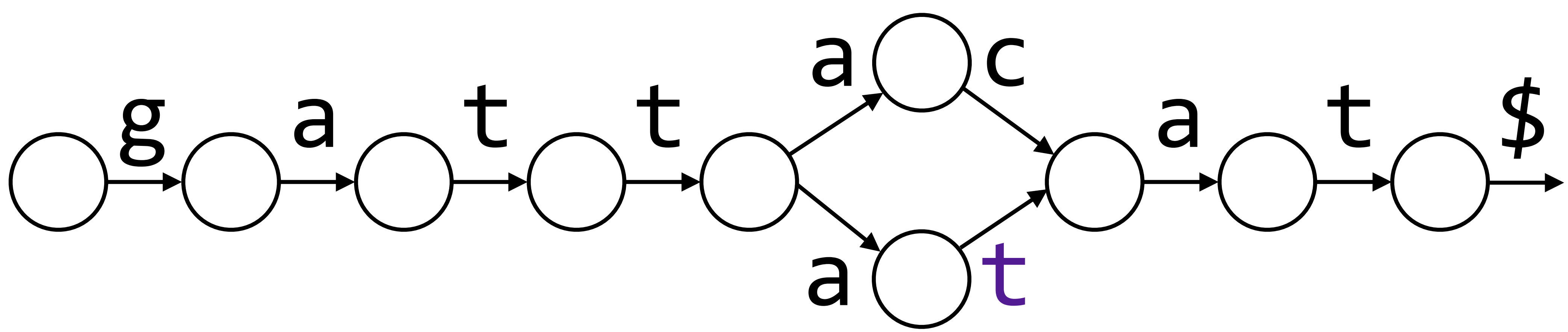
BWT: matching

For some graphs, total order exists



For others, not (but we can “fix" them sometimes)



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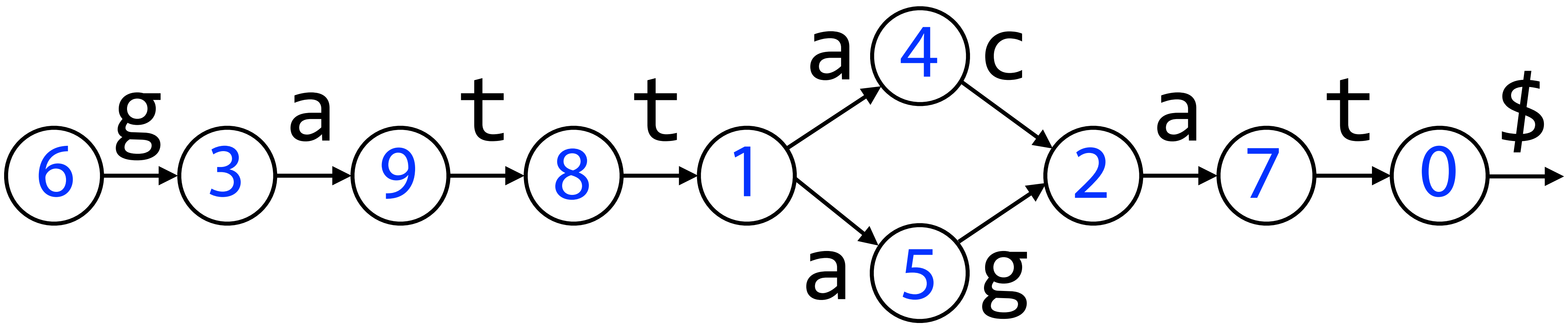

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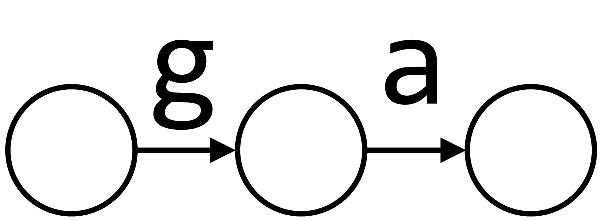

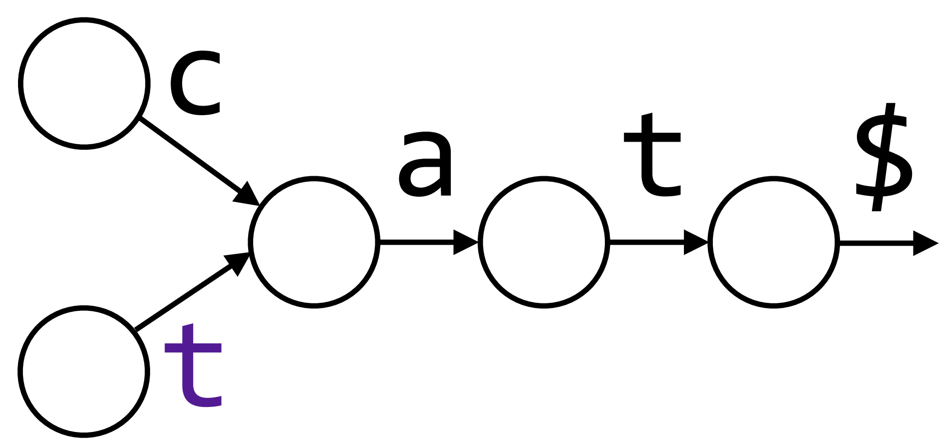

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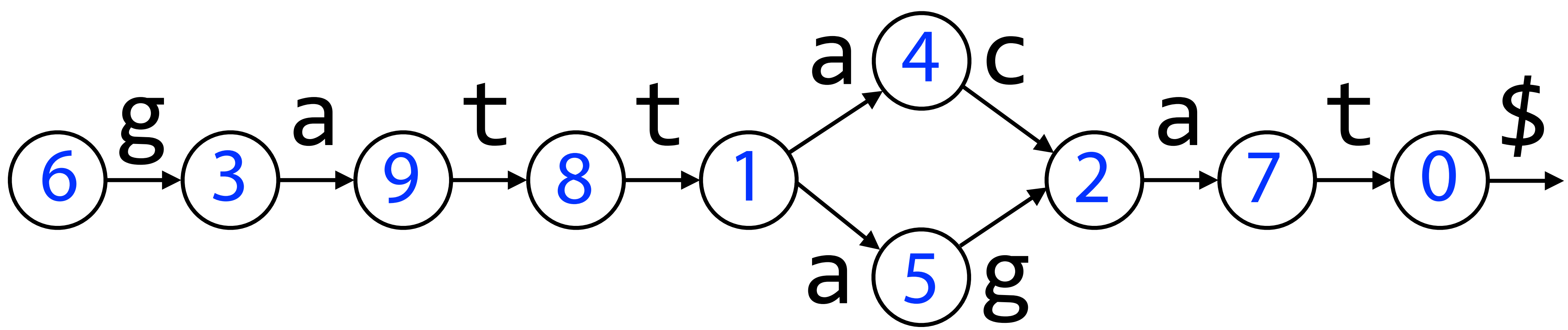

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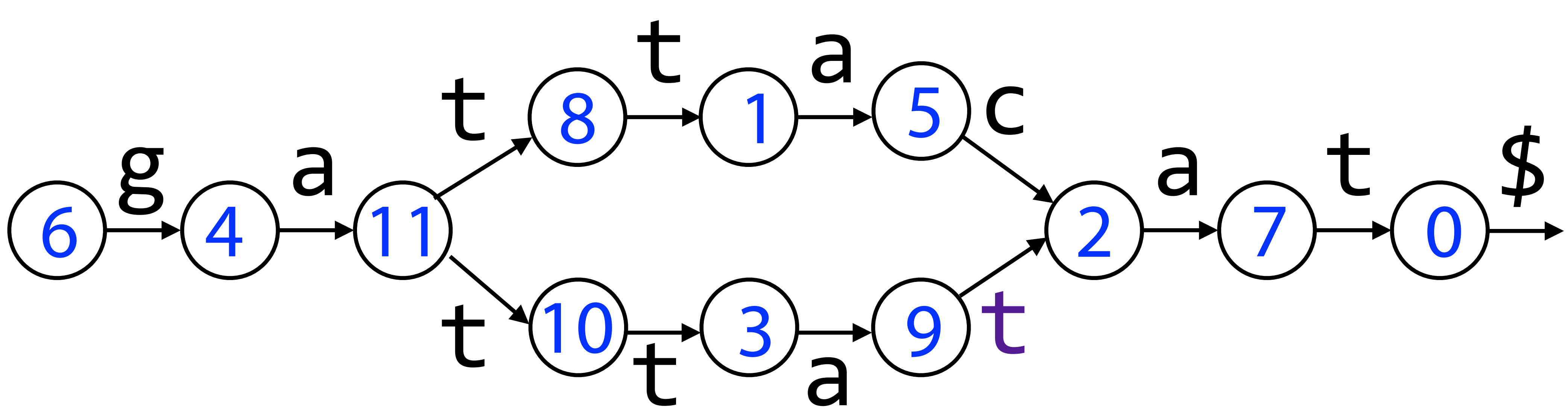

Wheeler graphs

Which graphs does it work for?

Wheeler graphs

An edge-labeled directed graph is a ***Wheeler Graph*** if nodes can be ordered such that:

Wheeler graphs

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(1) 0 in-degree nodes come before others

For all pairs of edges $e = (u, v), e' = (u', v')$ labeled a, a' respectively, we have:

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(3) $$(a = a') \wedge (u < u') \implies v \leq v'.$$

Wheeler graphs

$$A \implies B$$ "A implies B"

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A	B	$A \implies B$

Wheeler graphs

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A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

When left-hand side (LHS) is true, RHS must be true

When LHS is false, RHS can be whatever it wants

Wheeler graphs

(2) For all pairs of edges $e = (u, v), e' = (u', v')$ labeled a, a': $a \prec a' \implies v < v'$

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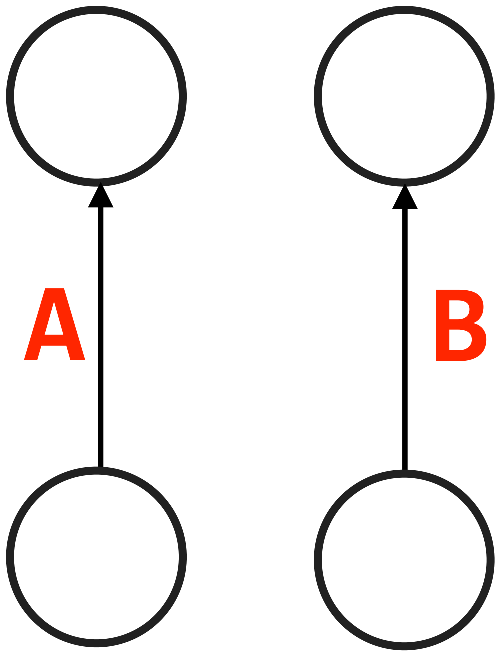

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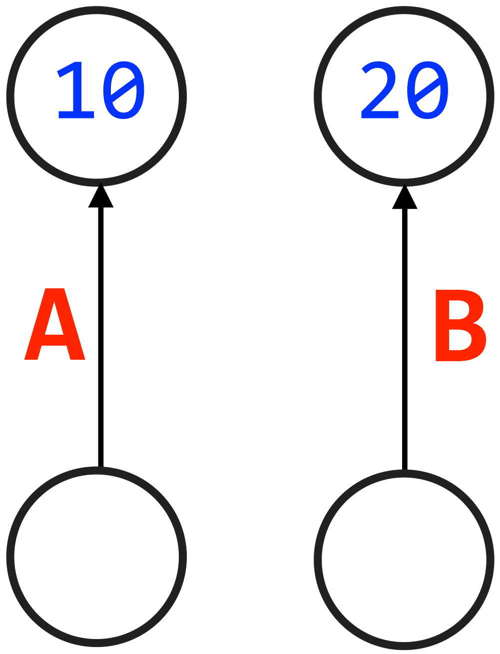

If labels differ, destination of smaller-label edge comes before destination of larger-label edge

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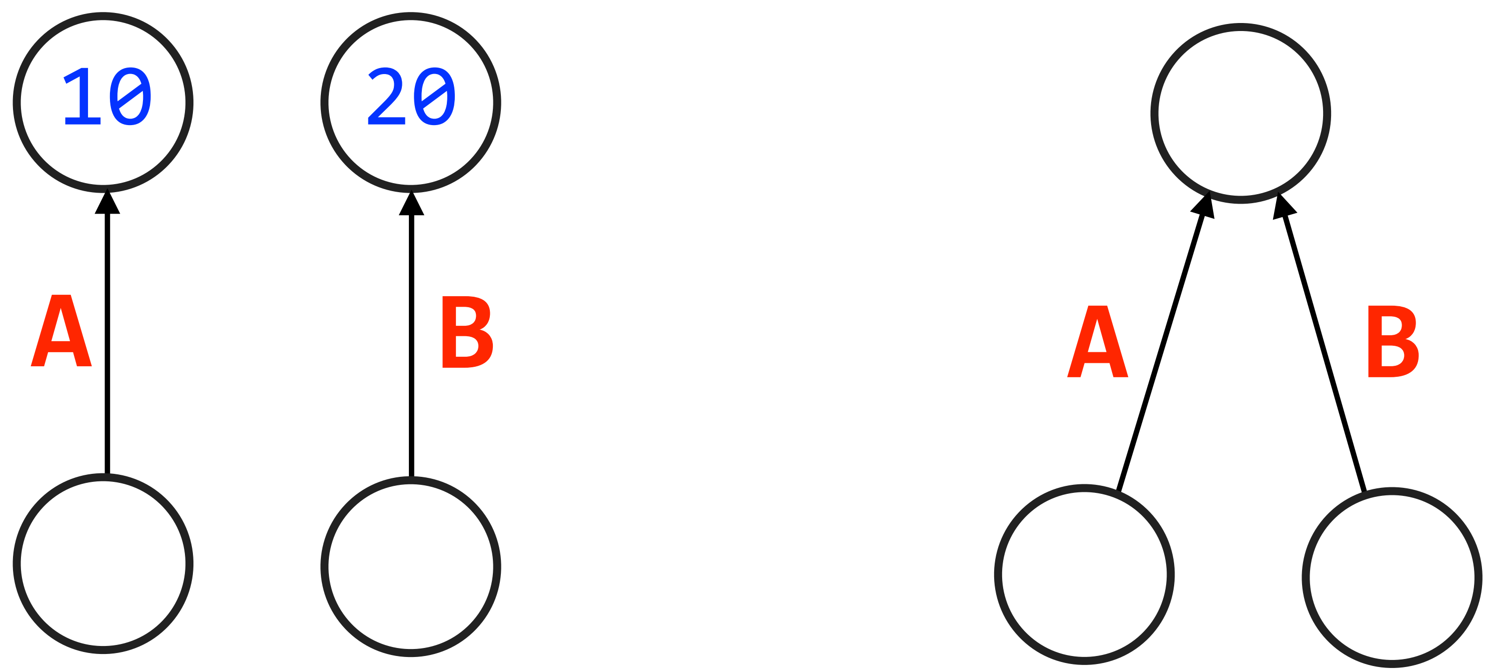

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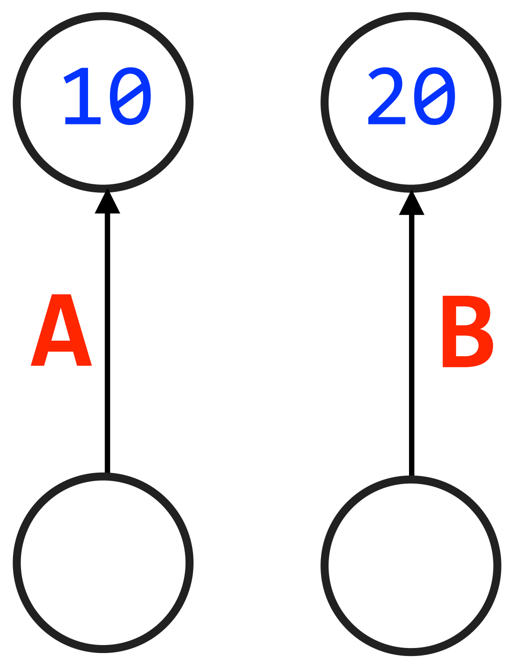

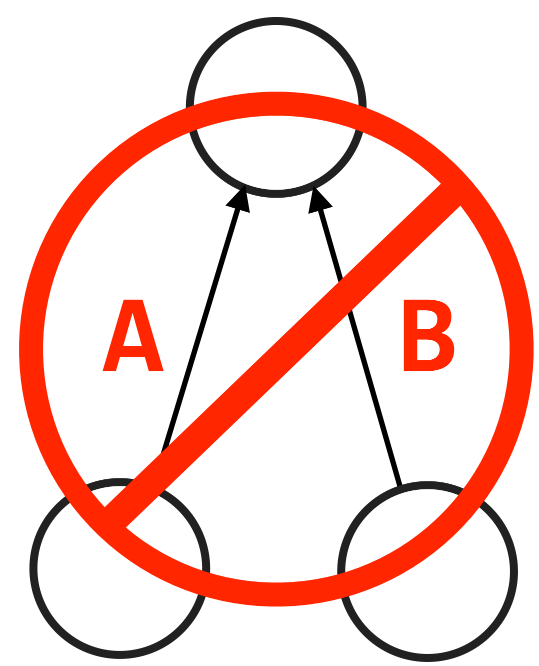

If labels differ, destination of smaller-label edge comes before destination of larger-label edge

Corollary: cannot have 2 incoming edges with different labels

Wheeler graphs

(3) For all pairs of edges $e = (u, v), e' = (u', v')$ labeled a, a': $(a = a') \wedge (u < u') \implies v \leq v'$.

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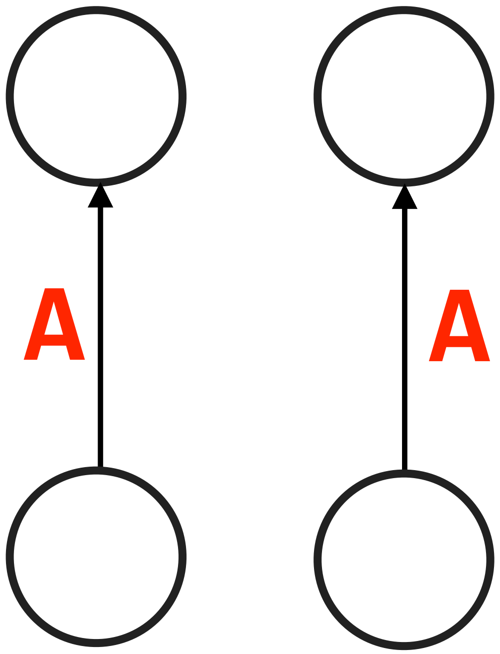

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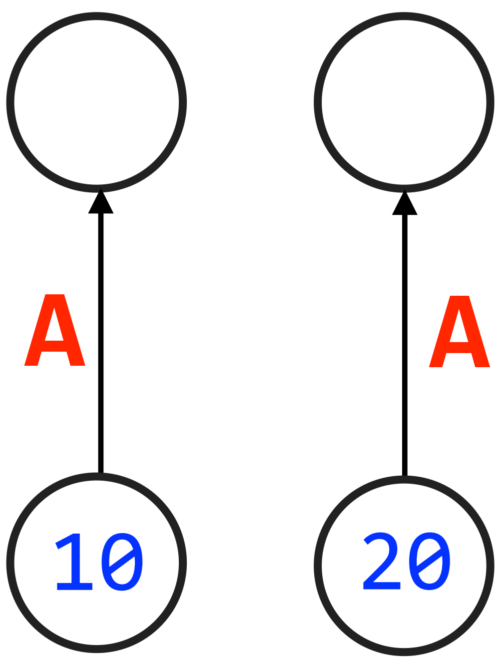

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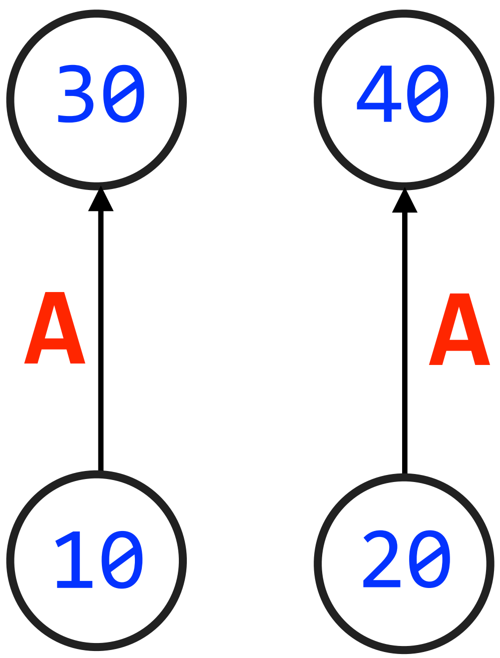

If labels match but sources differ, destination of the lower-source edge must not come after destination of the higher-source edge

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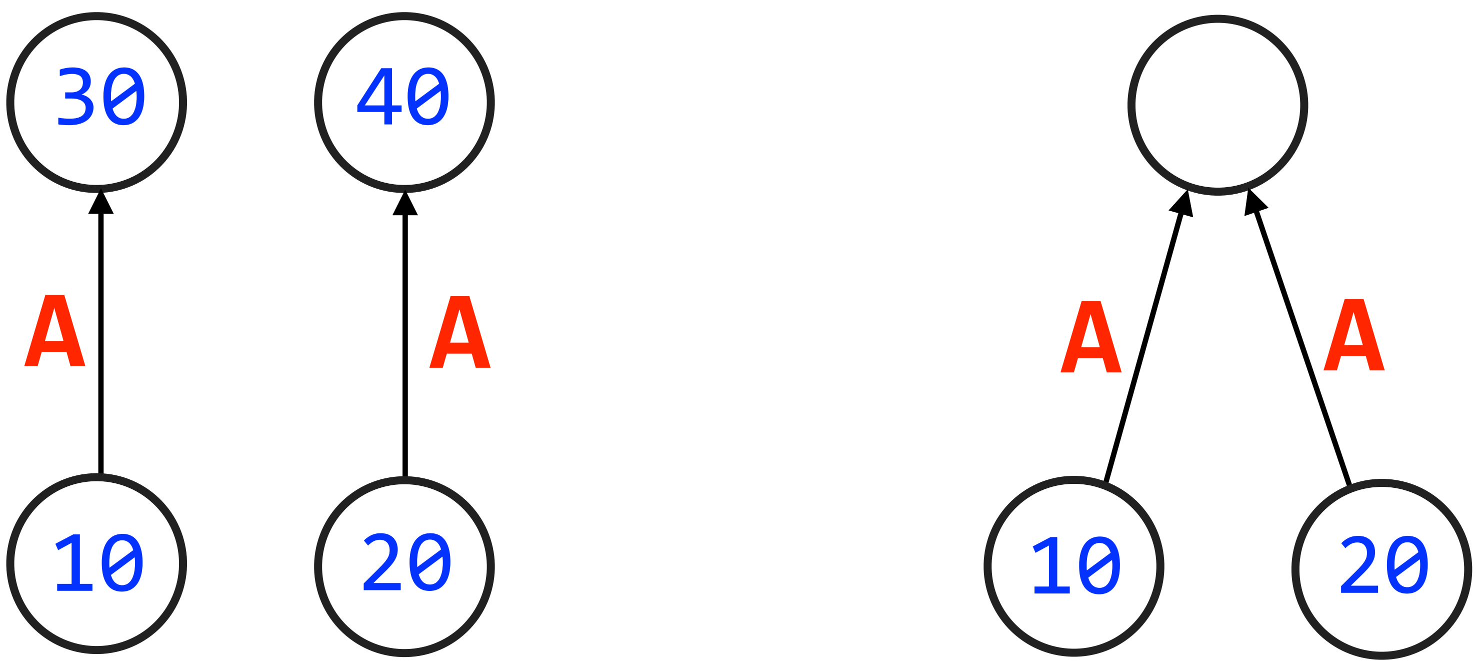

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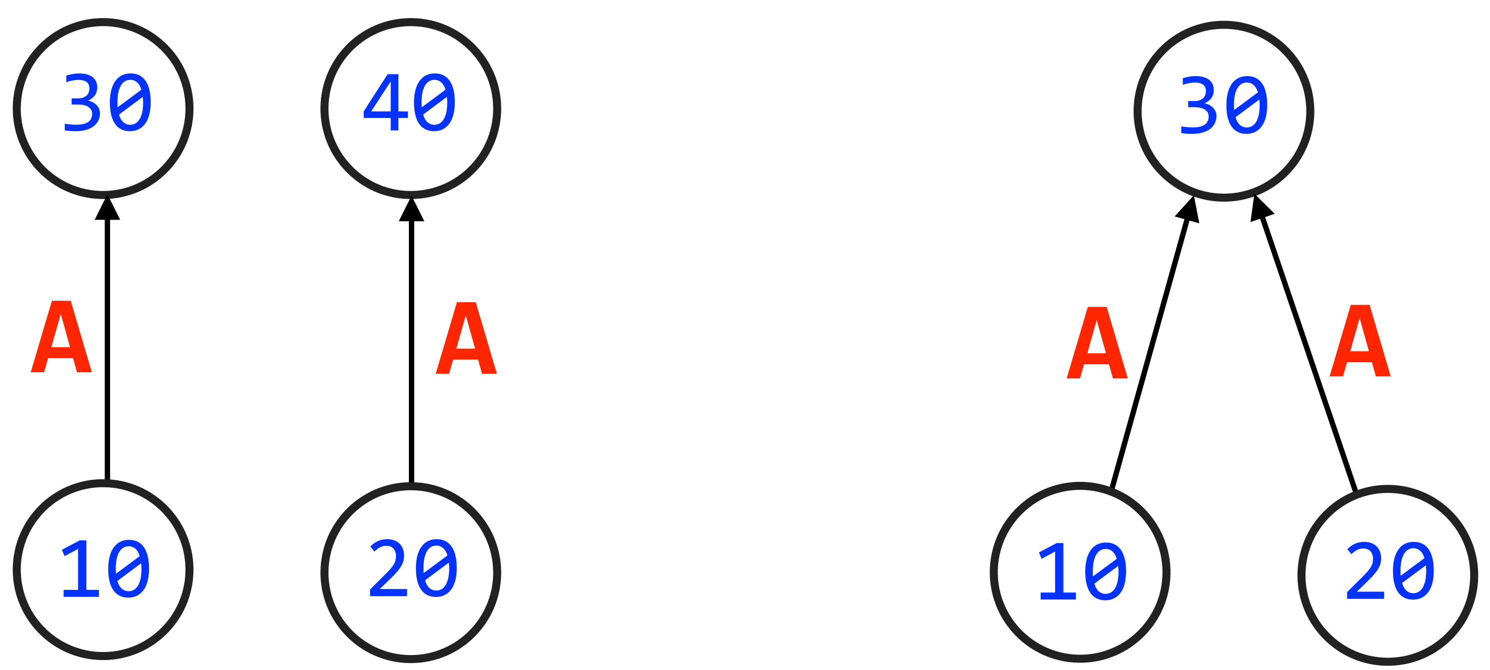

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0 in-degree nodes come before others $\quad (1)$

For all pairs of edges

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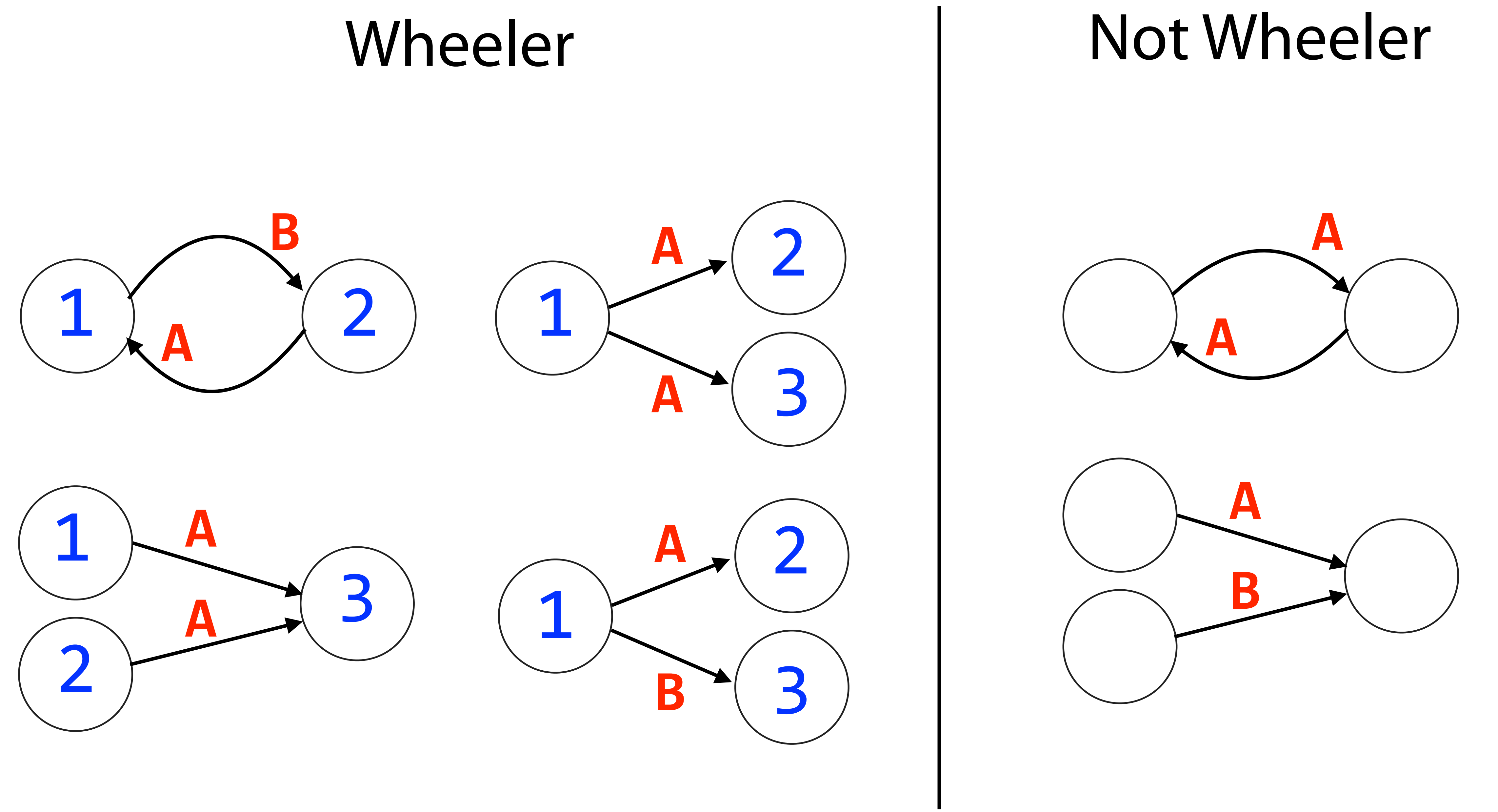

Wheeler graphs

Given an edge-labeled, directed multigraph, how hard is it to decide if it is a Wheeler Graph?

For given ordering, not hard

Properties are easily checked by looping over nodes, edges, pairs of edges

Exists an order over nodes s.t.:

0 in-degree nodes come before others (1)

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Wheeler graphs

Given an edge-labeled, directed multigraph, how hard is it to decide if it is a Wheeler Graph?

From scratch, it's **NP complete**

Related problems also hard to solve / approximate

D Gibney & SV Thankachan, "On the Hardness and Inapproximability of Recognizing Wheeler Graphs." 27th Annual European Symposium on Algorithms (ESA 2019), pp51:1--51:16

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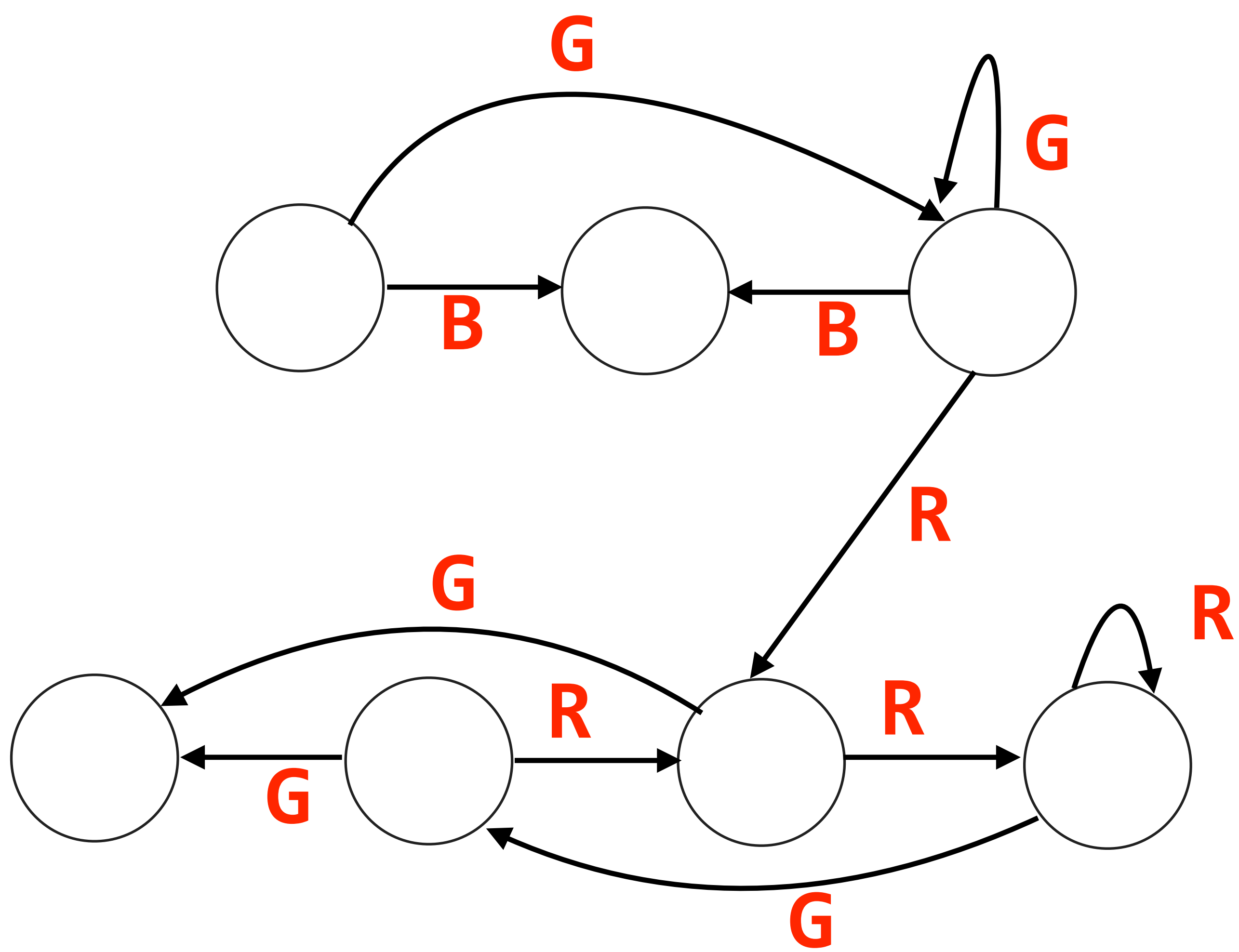
Is this a wheeler graph?



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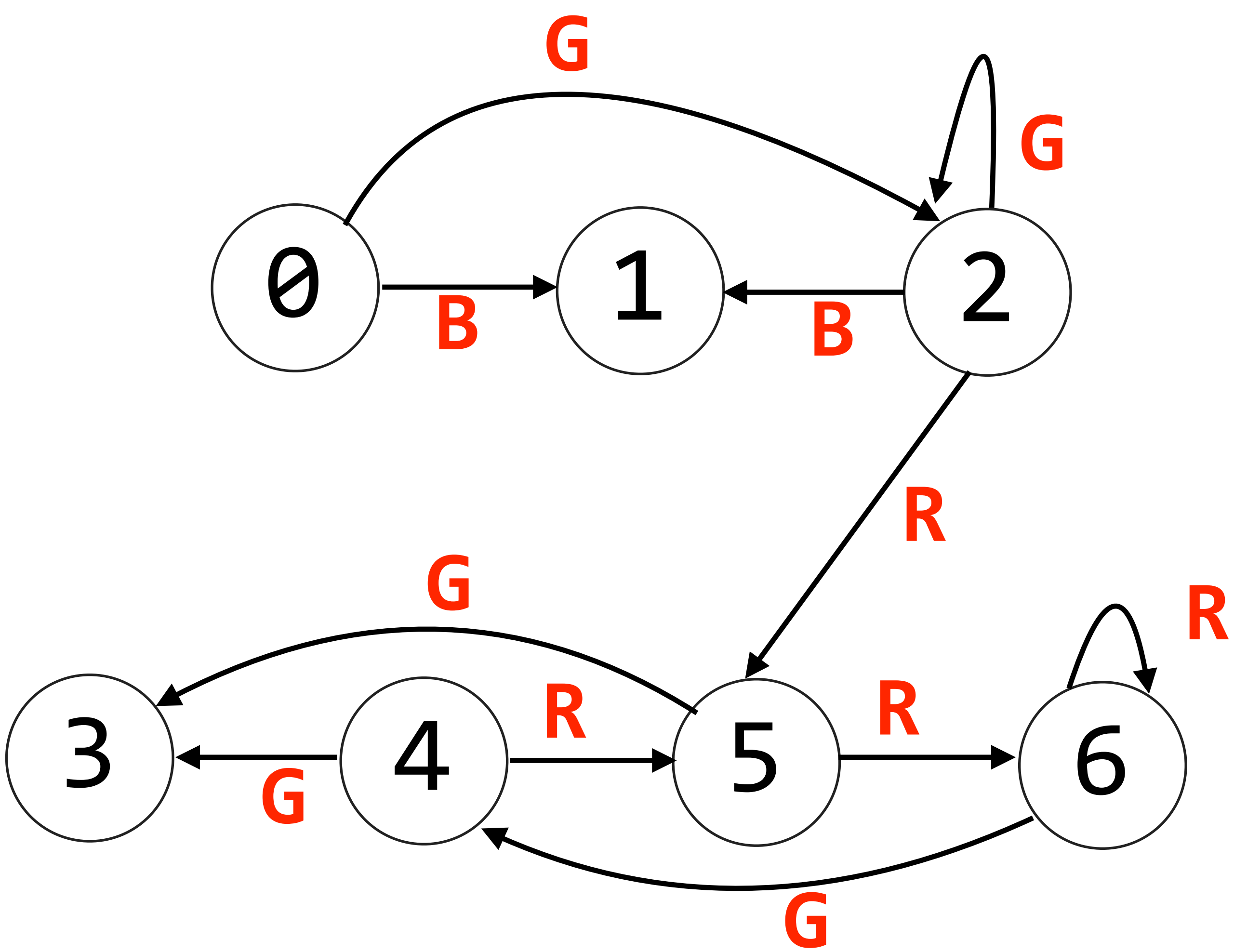

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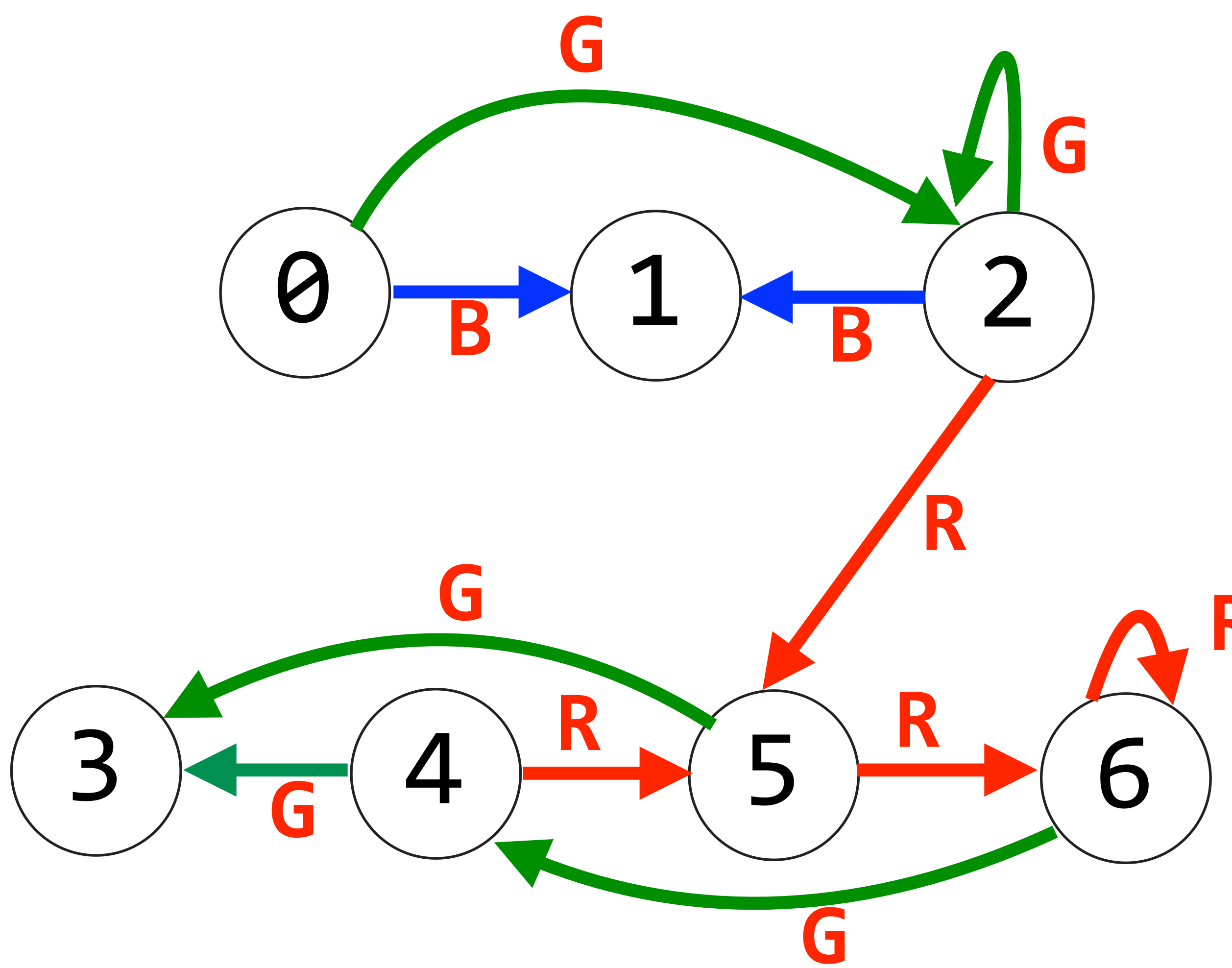

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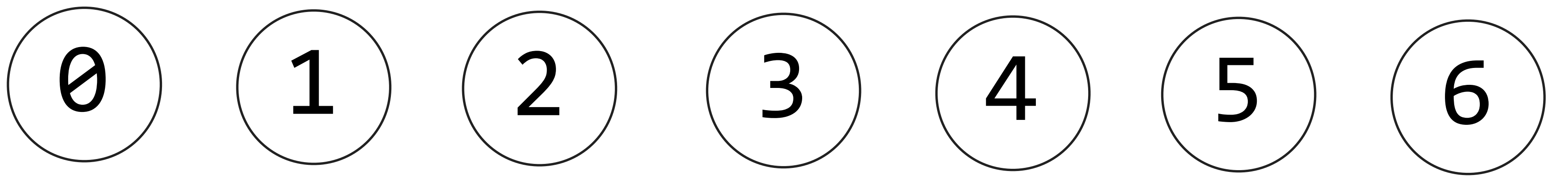

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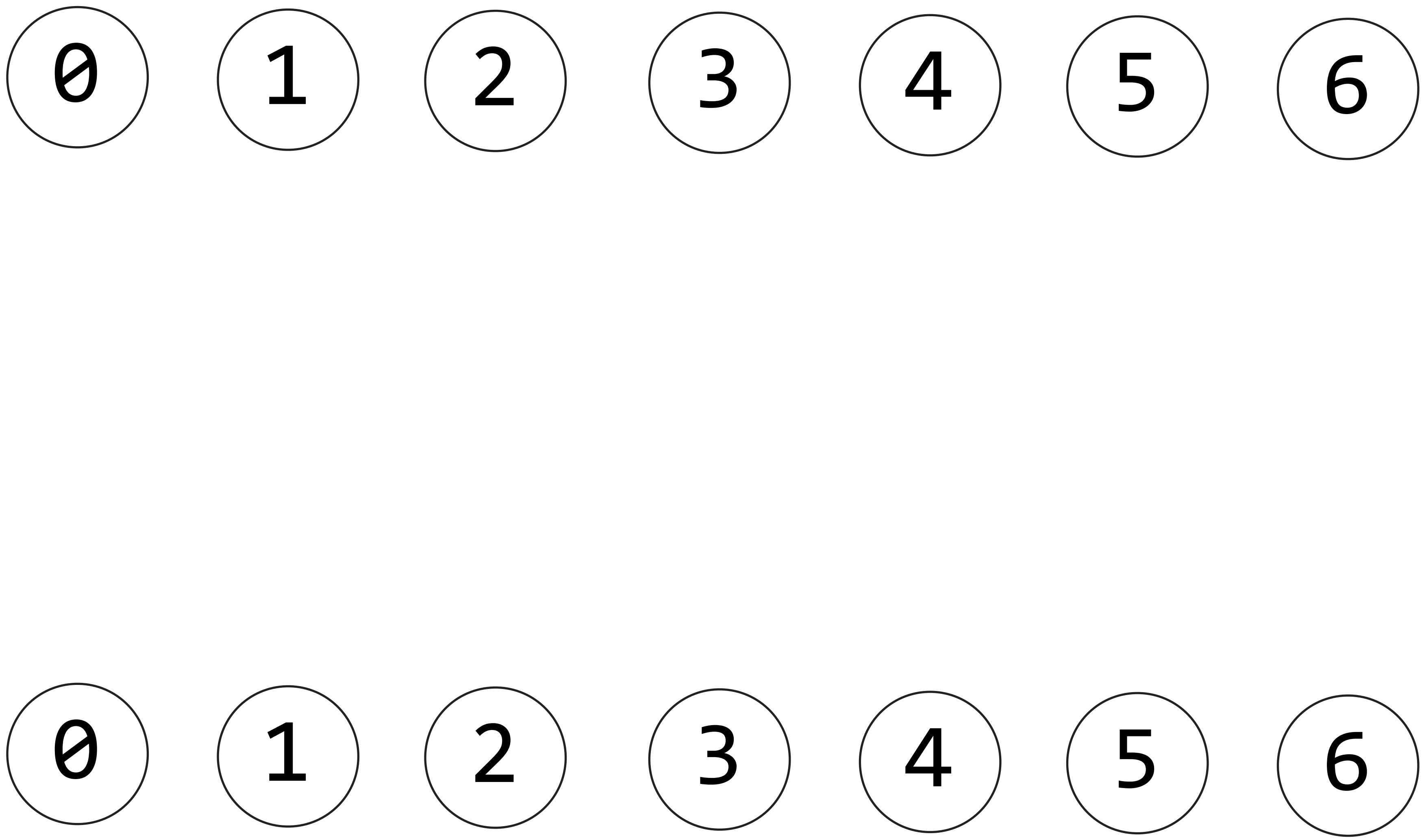

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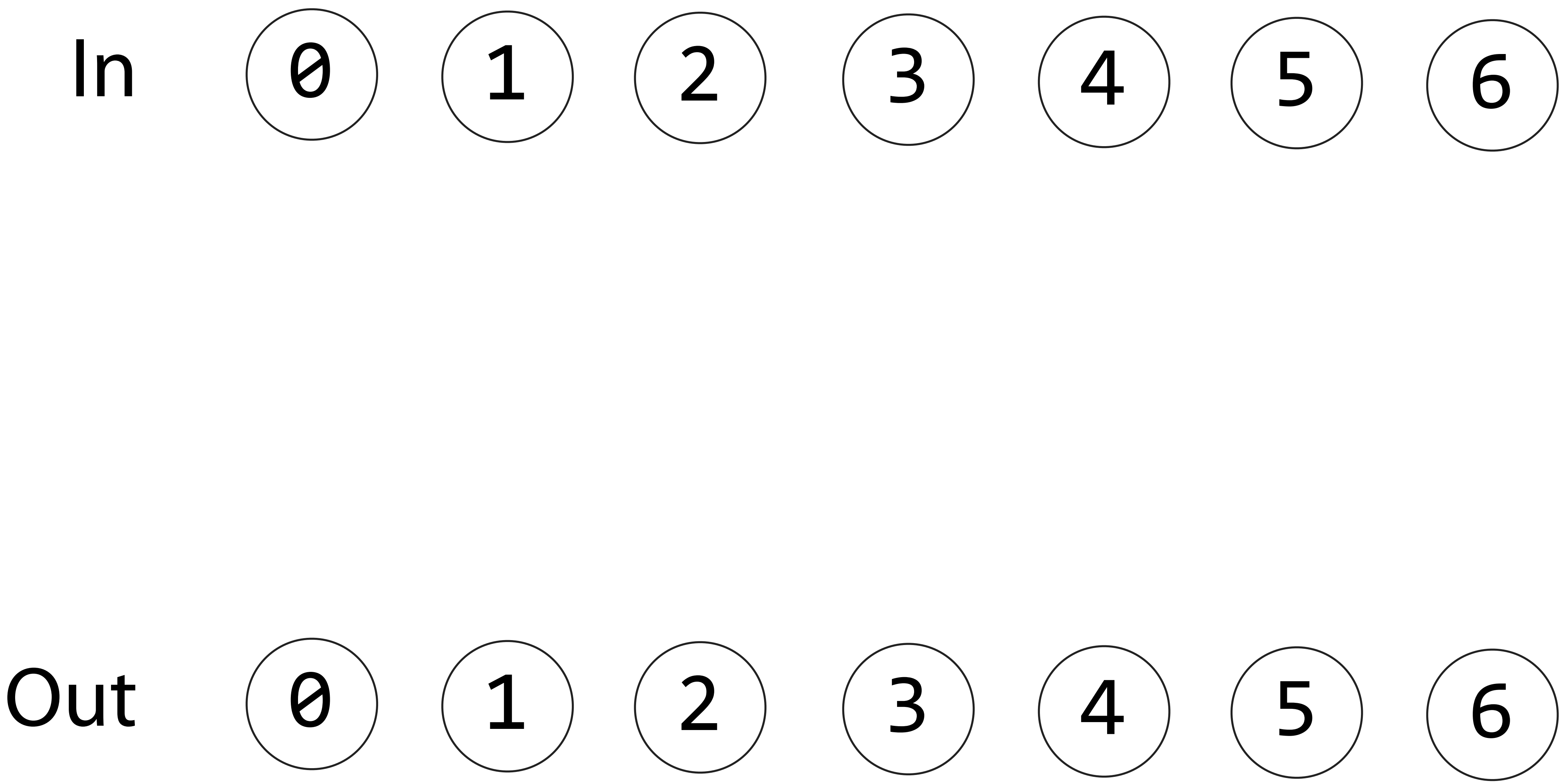

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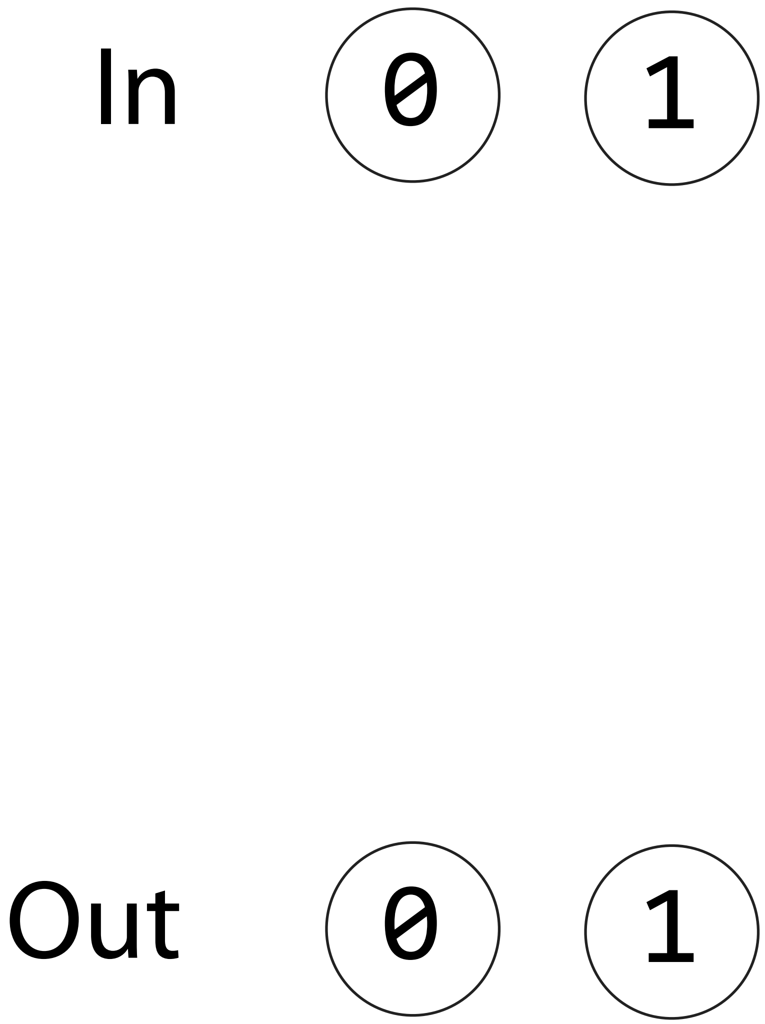
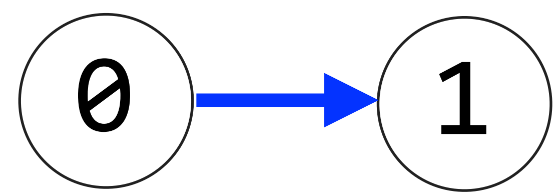


Original:



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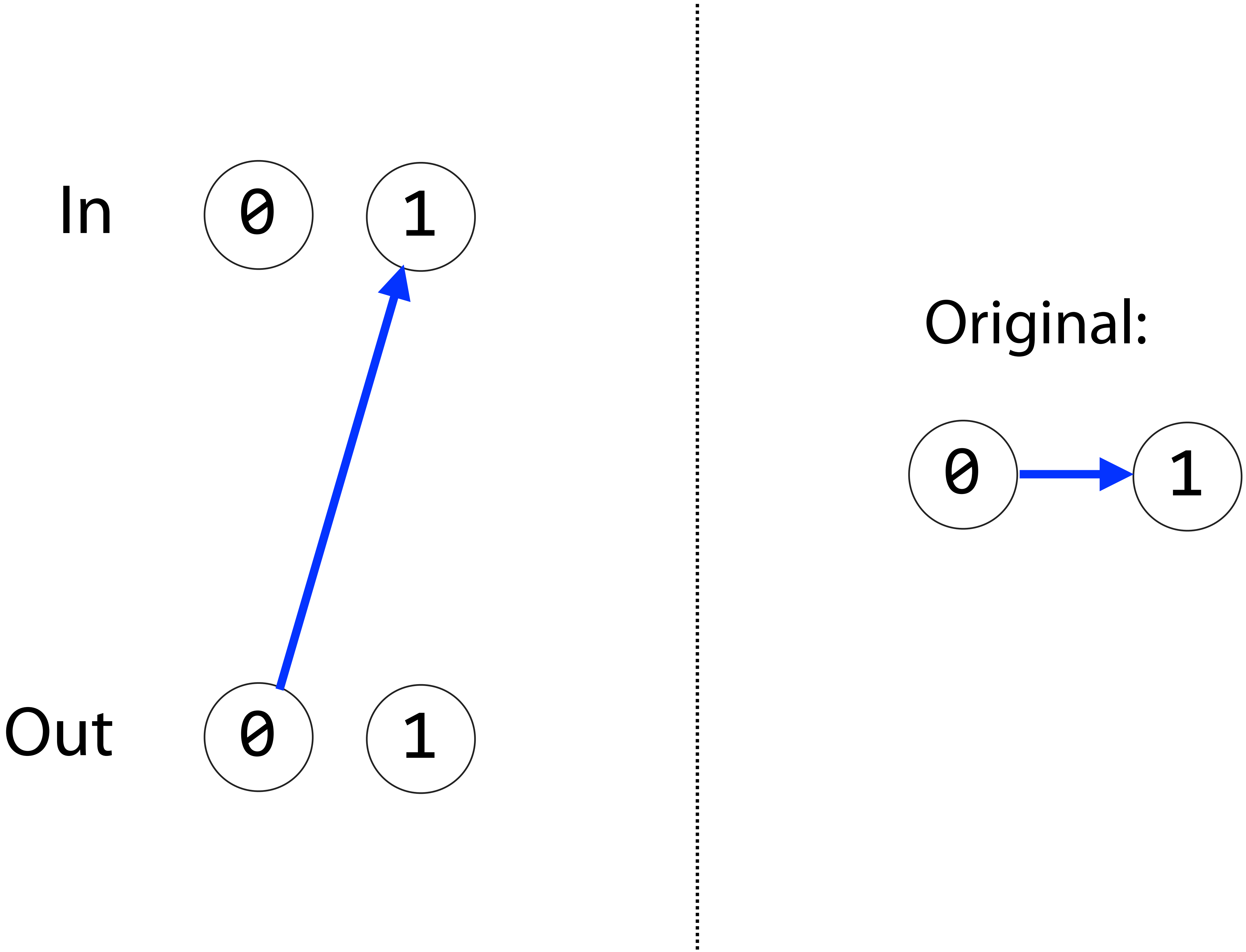

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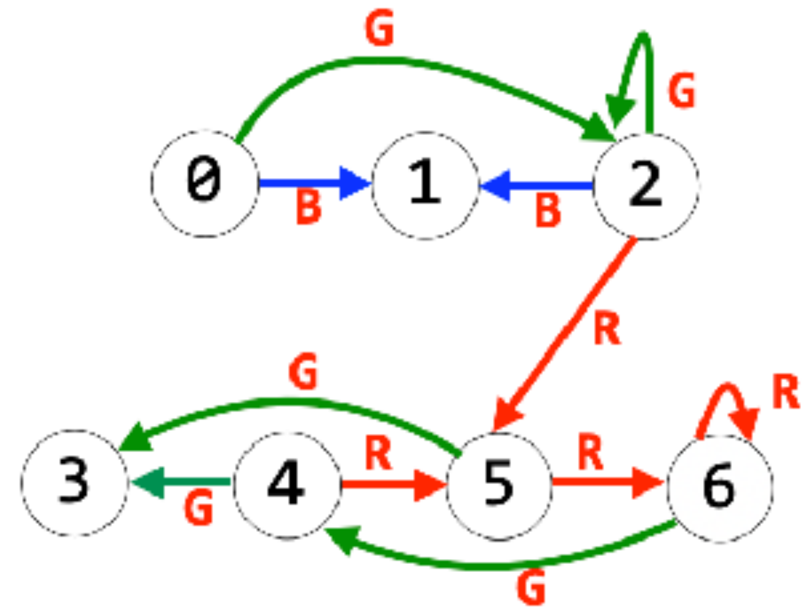

In 0 1 2 3 4 5 6

Out 0 1 2 3 4 5 6

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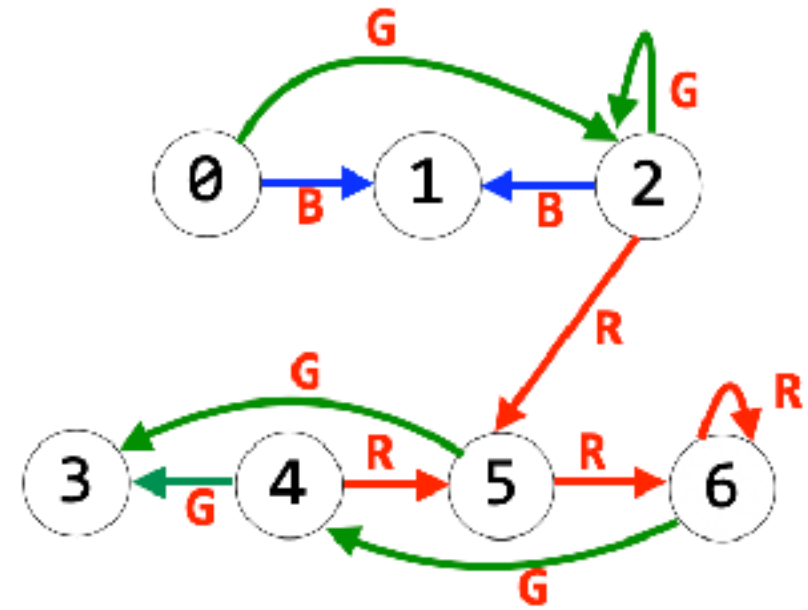
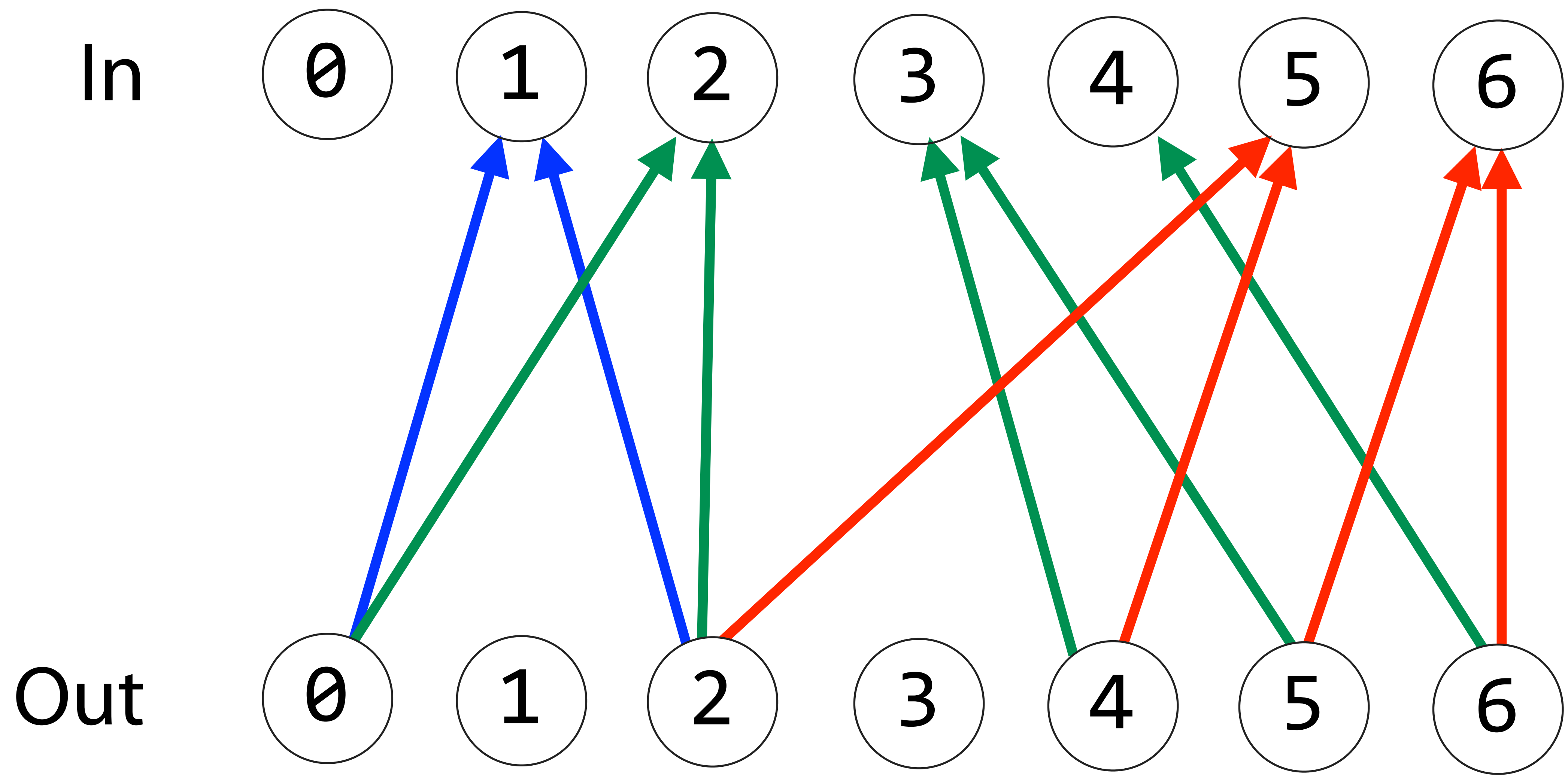
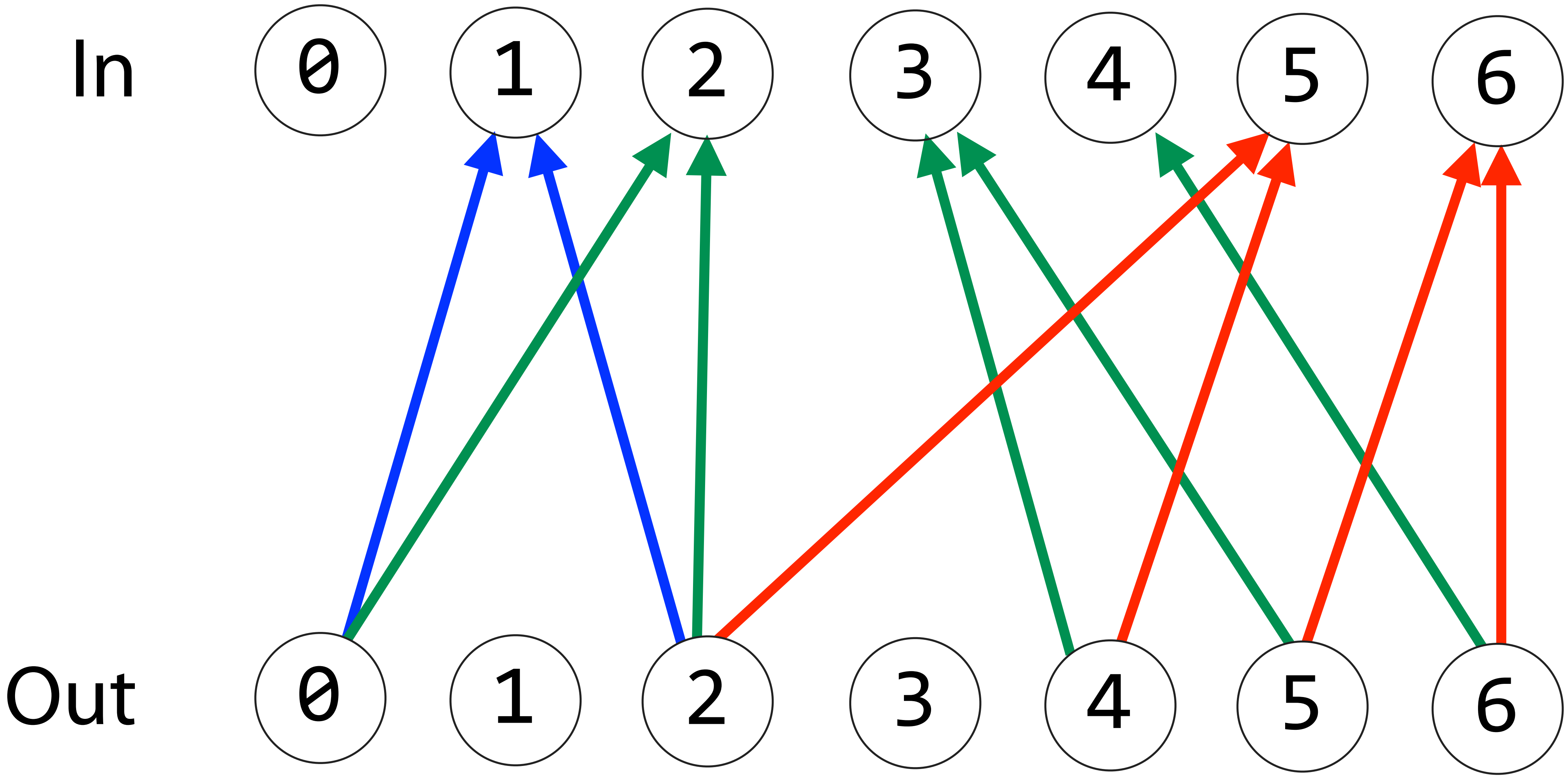




Wheeler graphs

$$a \prec a' \implies v < v'$$



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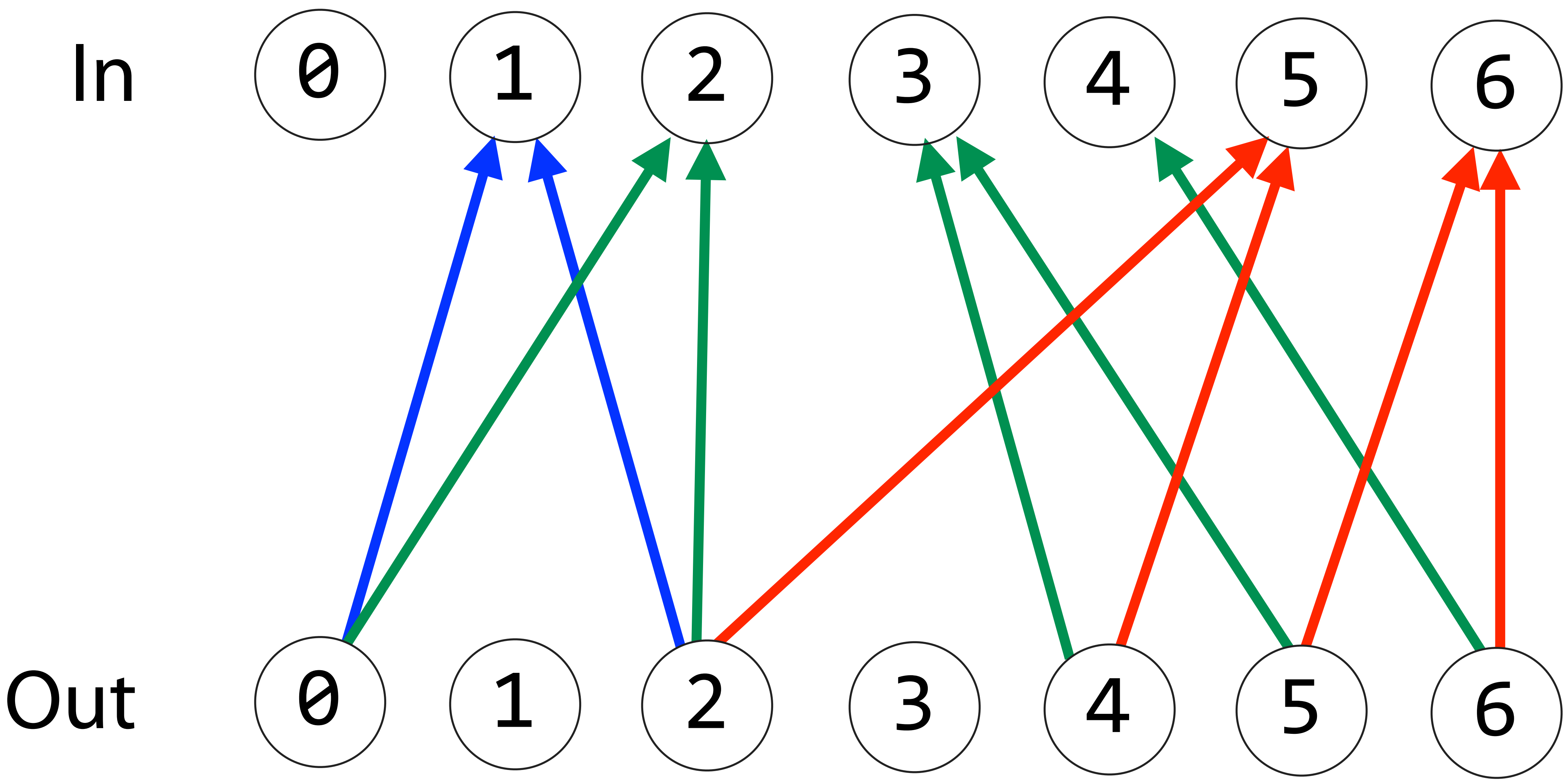
$$a \prec a' \Longrightarrow \; v < v'$$

Blue destinations before green destinations before red



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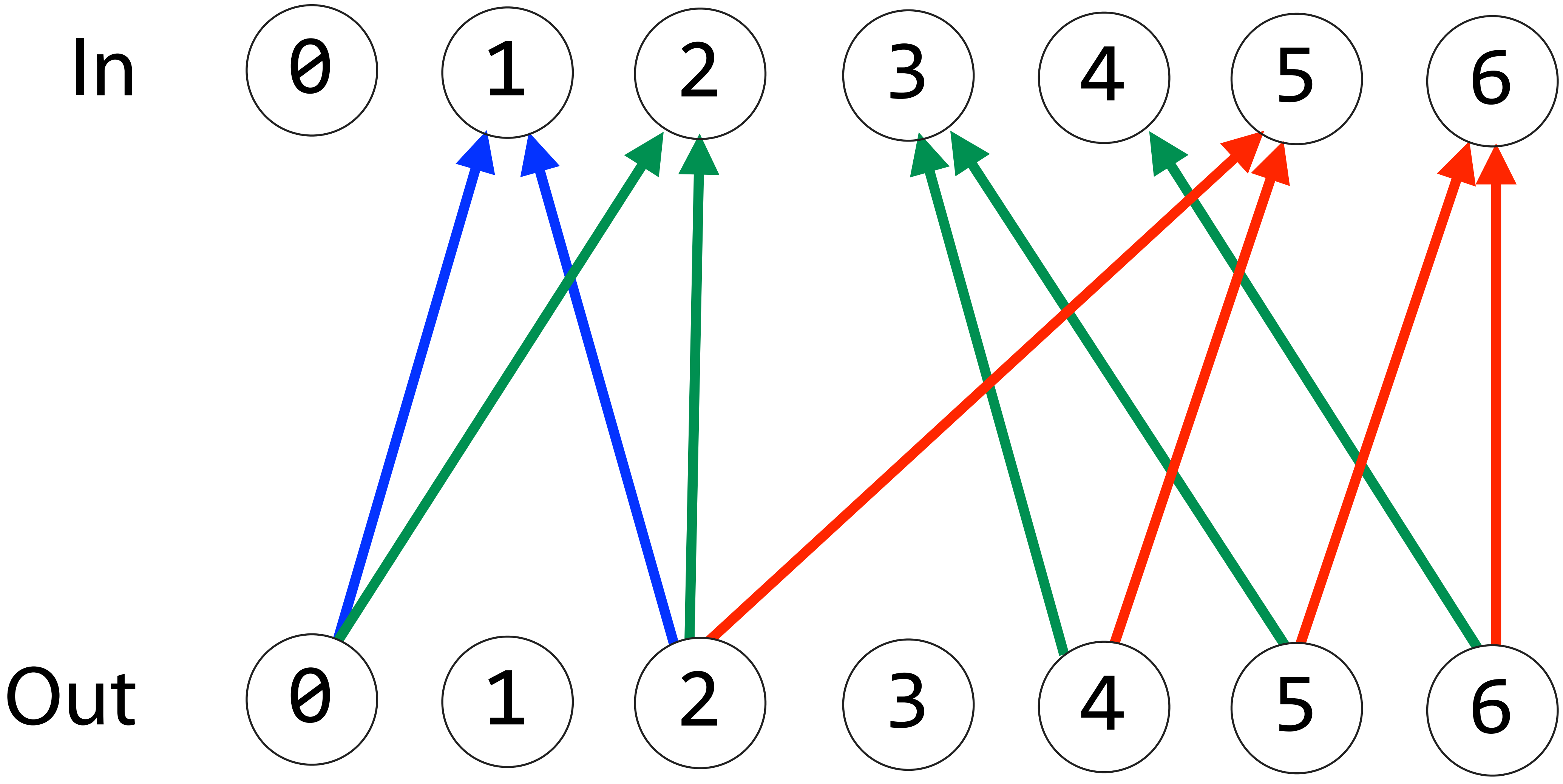

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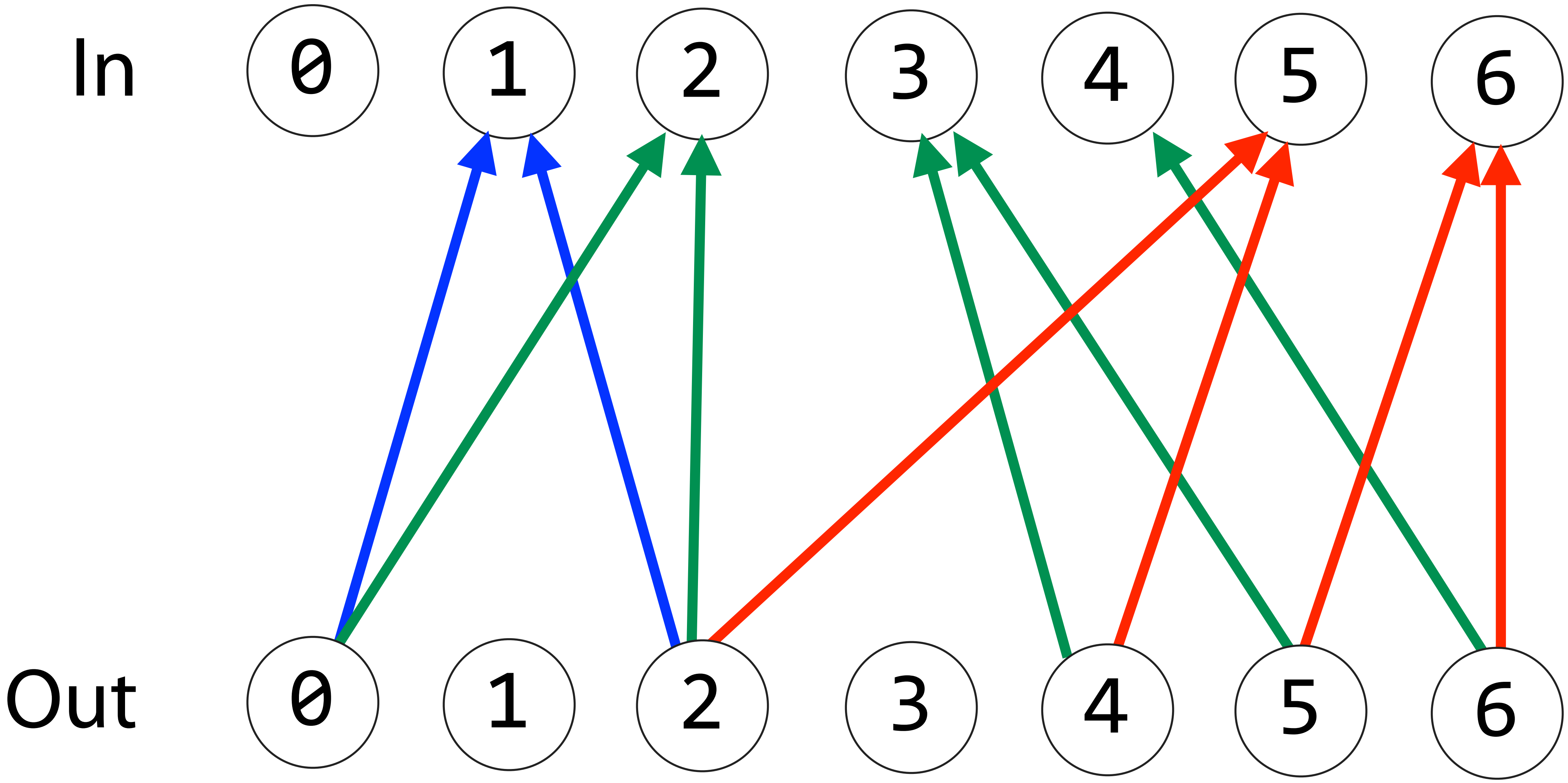

$$(a = a') \wedge (u < u') \implies v \leq v'$$

No same-color edges cross each other

Wheeler graphs

0 in-degree nodes come before others (1)

For all pairs of edges

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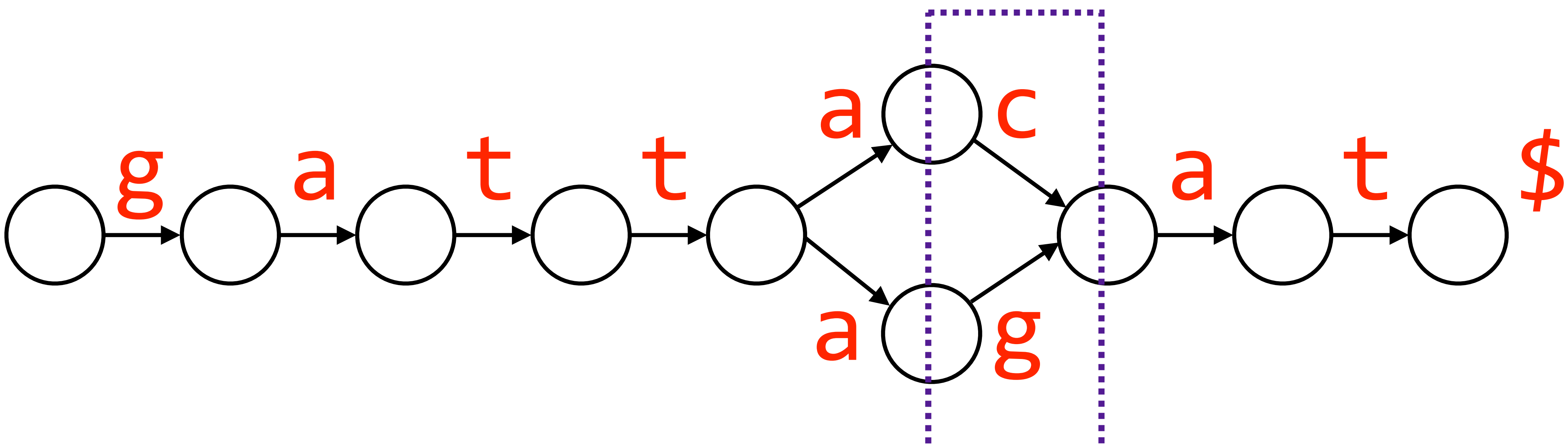
$$(a = a') \wedge (u < u') \implies v \leq v' \quad (3)$$

Is this a Wheeler Graph? **No**



$a \prec a'$ but $v = v'$ (2) cannot hold

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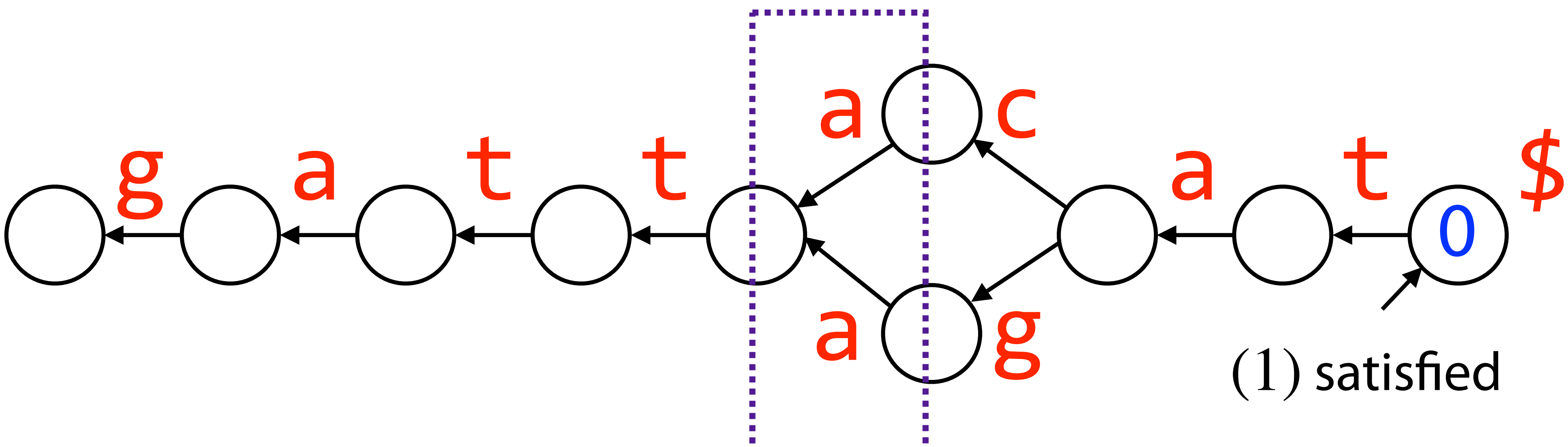
What if we flip edges to follow the direction of matching?



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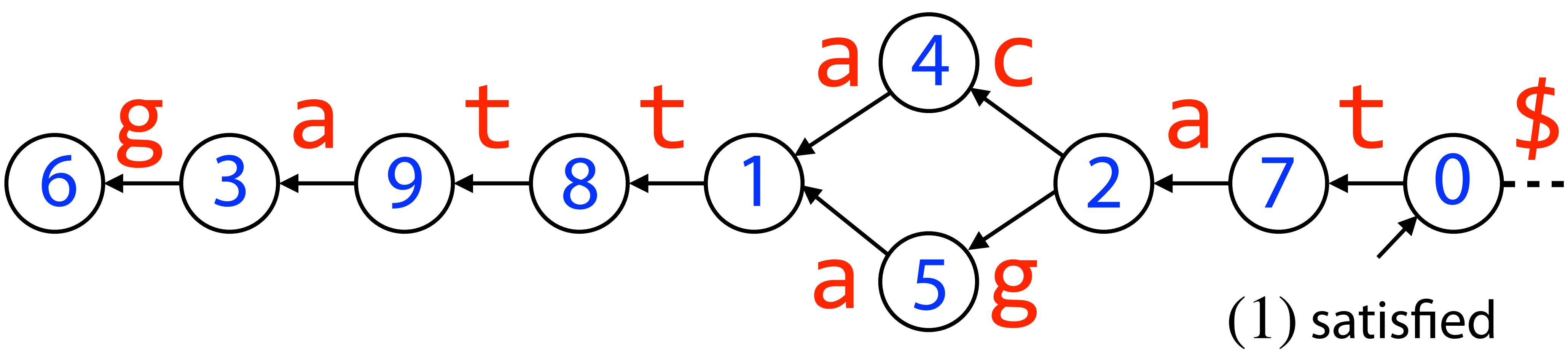
Successors of edges labeled:



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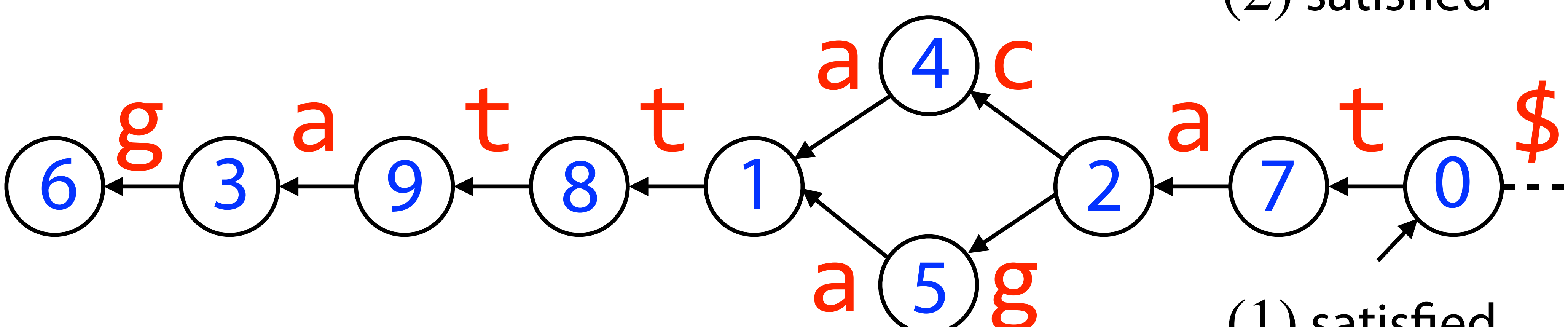
Successors of edges labeled: a : $\{1, 2, 3\}$ g : $\{5, 6\}$ c : $\{4\}$ t : $\{7, 8, 9\}$

(2) satisfied



(1) satisfied

Exercise: prove (3) is satisfied for all pairs of edges