

# On the Intersection of Context-Free and Regular Languages

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## Abstract

The Bar-Hillel construction is a classic result in formal language theory. It shows, by construction, that the intersection between a context-free language and a regular language is itself context-free. However, neither its original formulation (Bar-Hillel et al., 1961) nor its weighted extension (Nederhof and Satta, 2003) can handle automata with  $\varepsilon$ -arcs. In this short note, we generalize the Bar-Hillel construction to correctly compute the intersection even when the automaton contains  $\varepsilon$ -arcs. We further prove that our generalized construction leads to a grammar that encodes the structure of both the input automaton and grammar while retaining the asymptotic size of the original construction.

## 1 Introduction

Bar-Hillel et al.’s (1961) construction—together with its weighted generalization (Nederhof and Satta, 2003)—is a fundamental result in formal language theory. Given a weighted context-free grammar (WCFG)  $\mathcal{G}$  and a weighted finite-state automaton (WSFA)  $\mathcal{A}$ , the Bar-Hillel construction yields another WCFG  $\mathcal{G}_\cap$  whose language  $L(\mathcal{G}_\cap)$  is equal to the intersection of  $L(\mathcal{G})$  with  $L(\mathcal{A})$ . Importantly, the Bar-Hillel construction directly proves that weighted context-free languages are closed under intersection with weighted regular languages. The construction has later been extended to other formalisms (Maletti and Satta, 2009; Maletti, 2010; Dyer, 2010; Allauzen et al., 2014). Furthermore, the Bar-Hillel construction has seen applications in the computation of infix probabilities (Nederhof and Satta, 2011) and human sentence comprehension (Levy, 2008, 2011).

Unfortunately, Bar-Hillel et al.’s construction, as well as its weighted generalization by Nederhof and Satta, requires the input automaton  $\mathcal{A}$  to be  $\varepsilon$ -free.<sup>1</sup> Although any WSFA can be converted

<sup>1</sup>But they do not require the input grammar to be  $\varepsilon$ -free.

to a weakly equivalent<sup>2</sup>  $\varepsilon$ -free WSFA using well-known techniques (Mohri, 2001, 2002; Hanneforth and de la Higuera, 2010), such an approach adds an additional step of computation and does not, in general, maintain a bijection between derivations in  $\mathcal{G}_\cap$  and the Cartesian product of the derivations in  $\mathcal{G}$  and paths in  $\mathcal{A}$ . In other words,  $\mathcal{G}_\cap$  is *not* strongly equivalent<sup>3</sup> to the product of  $\mathcal{G}$  and  $\mathcal{A}$ .

In this note, we generalize the classic Bar-Hillel construction to the case where the automaton has  $\varepsilon$ -arcs.<sup>4</sup> Our new construction produces a WCFG  $\mathcal{G}_\cap$  that is strongly equivalent to the product of  $\mathcal{G}$  and  $\mathcal{A}$ . We further generalize the Bar-Hillel construction to work with arbitrary commutative semirings. Finally, we give an asymptotic bound on the size of the resulting grammar and a detailed proof of correctness in the appendix.<sup>5</sup>

## 2 Languages, Automata, and Grammars

As background, we now give formal definitions of weighted formal languages, finite-state automata, and context-free grammars.

### 2.1 Weighted Formal Languages

This paper concerns itself with transforms between devices for generating weighted formal languages.

**Definition 1.** *Let  $\Sigma$  be an alphabet and  $\mathcal{W} = (A, \oplus, \otimes, \mathbf{0}, \mathbf{1})$  a semiring. Then a **weighted formal language**  $L : \Sigma^* \rightarrow A$  is a mapping from the free monoid over  $\Sigma$  to the set of weights  $A$ . Furthermore, the set  $\text{supp}(L) = \{s \in \Sigma^* \mid L(s) \neq \mathbf{0}\}$  is called the language’s **support**.*

Unweighted formal languages (e.g., Sipser, 2006; Hopcroft et al., 2006) are simply the special

<sup>2</sup>Two WSAs are said to be weakly equivalent if they represent the same weighted formal language.

<sup>3</sup>This is formally defined in Definition 5 and Theorem 1.

<sup>4</sup>The technique we adopt to cover  $\varepsilon$ -arcs is a generalization of Allauzen et al. (2010), who intersected two FSAs.

<sup>5</sup>Our implementation is available at <https://github.com/rycolab/bar-hillel>.

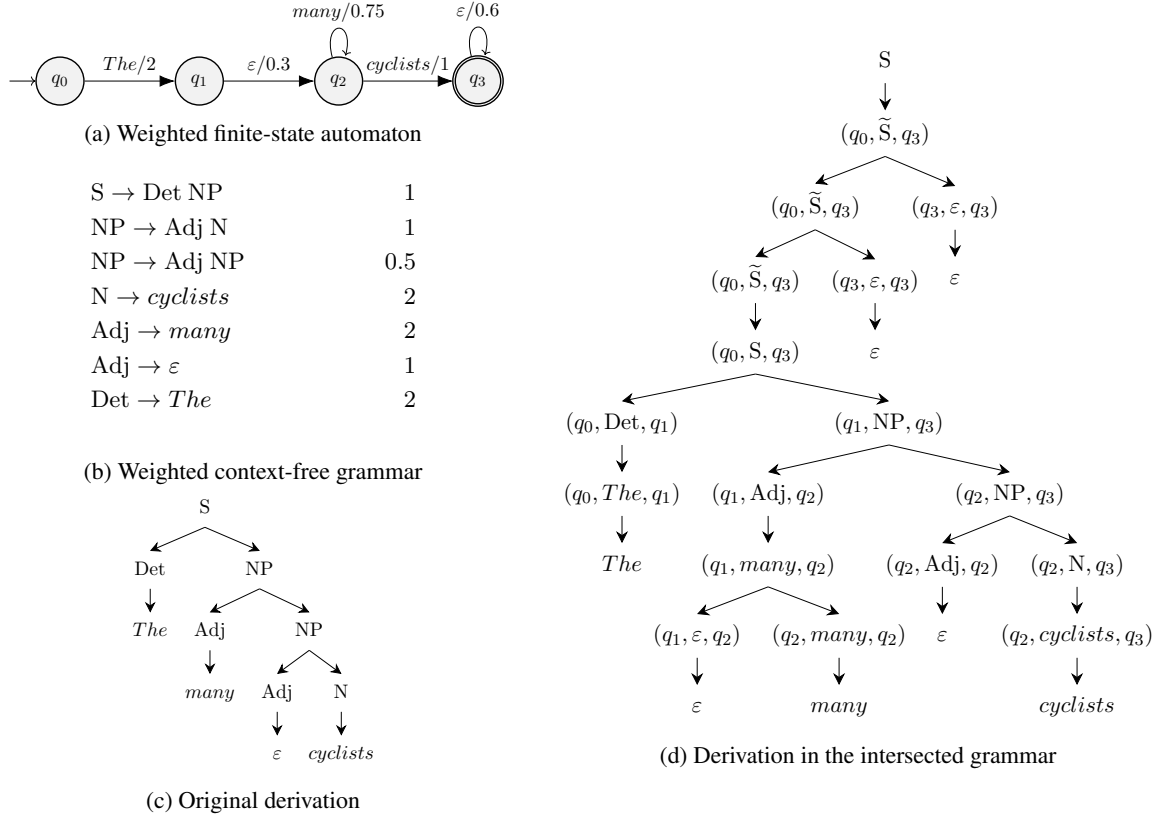


Figure 1: Example of a derivation in the grammar obtained as the intersection of the finite-state automaton (a) and the context-free grammar (b). The derivation tree (d) encodes the derivation tree (c) in the original grammar, and path  $q_0 \xrightarrow{The/2} q_1 \xrightarrow{\varepsilon/0.3} q_2 \xrightarrow{many/0.75} q_2 \xrightarrow{cyclists/1} q_3 \xrightarrow{\varepsilon/0.6} q_3 \xrightarrow{\varepsilon/0.6} q_3$ . To handle  $\varepsilon$ -arcs, we use rules from Eq. (7b) for  $\varepsilon$ -arcs appearing right before a final state, and rules from Eq. (7g) for  $\varepsilon$ -arcs appearing anywhere else.

case of Definition 1 where  $\mathcal{W}$  is the boolean semiring. In this note, we discuss algorithms for the computation of the intersection of two weighted formal languages.<sup>6</sup>

**Definition 2.** Let  $L_1$  and  $L_2$  be two weighted formal languages over the same alphabet  $\Sigma$  and the same semiring  $\mathcal{W}$ . The **intersection** of  $L_1$  with  $L_2$  is defined as the language:

$$(L_1 \cap L_2)(s) \stackrel{\text{def}}{=} L_1(s) \otimes L_2(s), \quad \forall s \in \Sigma^* \quad (1)$$

Specifically, this paper concerns itself with the special case of Definition 2 when  $L_1$  is a weighted context-free language (represented by a WCFG), and  $L_2$  is a weighted regular language (represented by a WFSA); we define these two formalisms in the subsequent sections.

## 2.2 Weighted Finite-State Automata

We now review the basics of weighted finite-state automata (WFSA), which provide a formalism to represent weighted regular languages.

<sup>6</sup>The intersection of two weighted formal languages is also called their Hadamard product (Droste et al., 2009).

**Definition 3.** A **weighted finite-state automaton**  $\mathcal{A}$  over a semiring  $\mathcal{W} = (A, \oplus, \otimes, \mathbf{0}, \mathbf{1})$  is a 6-tuple  $(\Sigma, Q, \delta, \lambda, \rho, \mathcal{W})$ . In this tuple,  $\Sigma$  is an alphabet,  $Q$  is a finite set of states,  $\delta \subseteq Q \times Q \times (\Sigma \cup \{\varepsilon\}) \times A$  is a finite multi-set of weighted arcs. Further,  $\lambda : Q \rightarrow A$  and  $\rho : Q \rightarrow A$  are the initial and final weight functions, respectively. We also define the sets  $I = \{q \mid q \in Q, \lambda(q) \neq \mathbf{0}\}$  and  $F = \{q \mid q \in Q, \rho(q) \neq \mathbf{0}\}$  for convenience.

We will represent an arc in  $\delta$  with the notation  $q_0 \xrightarrow{a/w} q_1$  where  $a \in \Sigma \cup \{\varepsilon\}$  and  $w \in A$ . A **path**  $\pi$  (of length  $N > 0$ ) is a sequence of arcs in  $\delta^*$  where the states of adjacent arcs are matched, i.e.

$$q_0 \xrightarrow{a_1/w_1} \dots q_{n-1} \xrightarrow{a_n/w_n} q_n \dots \xrightarrow{a_N/w_N} q_N \quad (2)$$

and where  $q_0 \in I$  and  $q_N \in F$ , i.e., the path starts in an initial state and ends in a final state. The path's **yield**, denoted  $\text{yield}(\pi)$ , is the concatenation  $a_1 a_2 \dots a_N$  of all of its input symbols; the path's **weight**, denoted  $w(\pi)$ , is the product:

$$w(\pi) = \lambda(q_0) \otimes \left( \bigotimes_{n=1}^N w_n \right) \otimes \rho(q_N) \quad (3)$$

We denote the set of all paths in  $\mathcal{A}$  as  $\mathcal{D}_{\mathcal{A}}$ , and the set of all paths with yield  $s$  as  $\mathcal{D}_{\mathcal{A}}(s)$ . Finally, we define the **language of an automaton** as the mapping  $L_{\mathcal{A}} : \Sigma^* \rightarrow A$  where we have  $L_{\mathcal{A}}(s) = \bigoplus_{\pi \in \mathcal{D}_{\mathcal{A}}(s)} w(\pi)$ . The set of languages that can be encoded by a WFSAs forms the class of **weighted regular languages**.

### 2.3 Weighted Context-Free Grammars

We now go over the necessary background on weighted context-free grammars (WCFGs).

**Definition 4.** A *weighted context-free grammar* is a tuple  $\mathfrak{G} = (\mathcal{N}, \Sigma, \mathcal{W}, S, \mathcal{P})$ , where  $\mathcal{N}$  is a non-empty set of non-terminal symbols,  $\Sigma$  is an alphabet of terminal symbols,  $\mathcal{W} = (A, \oplus, \otimes, \mathbf{0}, \mathbf{1})$  is a semiring,  $S \in \mathcal{N}$  is a distinguished start symbol, and  $\mathcal{P}$  is a set of production rules. Each rule  $p \in \mathcal{P}$  is of the form  $X \xrightarrow{w} \alpha$ , with  $X \in \mathcal{N}$ ,  $\alpha \in (\Sigma \cup \mathcal{N})^*$  and  $w \in A$ .

Given two strings  $\alpha, \beta \in (\Sigma \cup \mathcal{N})^*$ , we write  $\alpha \xrightarrow{p}_L \beta$  if and only if we can express  $\alpha = z X \delta$  and  $\beta = z \gamma \delta$  where  $z \in \Sigma^*$  and  $p \in \mathcal{P}$  is the rule  $X \xrightarrow{w} \gamma$ . A **derivation**  $d$  (more precisely, a leftmost derivation) is a sequence  $\alpha_0, \dots, \alpha_N$  with  $N > 0$ ,  $\alpha_0 = S$ , and  $\alpha_N \in \Sigma^*$ , such that for all  $0 < n \leq N$ , we have  $\alpha_{n-1} \xrightarrow{p_n}_L \alpha_n$  for some (necessarily unique)  $p_n \in \mathcal{P}$ . The derivation's **yield**,  $\text{yield}(d)$ , is  $\alpha_N$ , and its weight,  $w(d)$ , is  $w(p_1) \otimes \dots \otimes w(p_N)$ . We denote the set of derivations under a grammar  $\mathfrak{G}$  by  $\mathcal{D}_{\mathfrak{G}}$  and the set of all derivations with yield  $s$  by  $\mathcal{D}_{\mathfrak{G}}(s)$ . Finally, we define the **language of a grammar** as  $L_{\mathfrak{G}}$  where

$$L_{\mathfrak{G}}(s) \stackrel{\text{def}}{=} \bigoplus_{d \in \mathcal{D}_{\mathfrak{G}}(s)} w(d), \quad \forall s \in \Sigma^* \quad (4)$$

The languages that can be encoded by a WCFG are known as **weighted context-free languages**.

### 3 Generalizing Bar-Hillel

Given any context-free grammar (CFG)  $\mathfrak{G}$  and finite-state automata (FSA)  $\mathcal{A}$ , Bar-Hillel et al. (1961) showed how to construct a CFG  $\mathfrak{G}_{\cap}$  such that  $L_{\mathfrak{G}_{\cap}} = L_{\mathfrak{G}} \cap L_{\mathcal{A}}$ . Later, Nederhof and Satta (2003) generalized Bar-Hillel's construction to work on a WCFG and WFSAs, yielding a WCFG. While they focused on the real semiring, their construction actually works for any commutative semiring. The rules of  $\mathfrak{G}_{\cap}$  are given in Construction 1 (in Fig. 2). However, neither of these versions correctly computes the intersection when the

WFSAs (or FSAs) contains  $\varepsilon$ -arcs. Yet, in several applications—such as modeling noisy inputs for human sentence comprehension (Levy, 2008, 2011)—we may be interested in dealing with automata that contain  $\varepsilon$ -arcs. A naive application of the construction would ignore paths in the WFSAs that contain  $\varepsilon$ -arcs. The problem may be sidestepped by transforming the WFSAs into an  $\varepsilon$ -free version before applying the construction; this, however, would not allow us to identify the paths in the input WFSAs that yield a target string in the intersection grammar.

**The problem with  $\varepsilon$ -arcs.** Before proposing our solution, we illustrate how the previous construction fails in the case of  $\varepsilon$ -arcs. For a WFSAs  $\mathcal{A}$  and a WCFG  $\mathfrak{G}$ , their intersection  $\mathfrak{G}_{\cap}$  is defined by the rules of Fig. 2.<sup>7</sup> Rules defined by Eq. (5b) represent rules in  $\mathfrak{G}$ . When the intersection is non-empty they successfully match with rules defined by Eq. (5d), which encode arcs in  $\mathcal{A}$ , to form a derivation. Finally rules defined by Eq. (5c) handle  $\varepsilon$ -productions in  $\mathfrak{G}$  as a special case. Now consider the automaton and the grammar in Fig. 1, both the automaton and the grammar assign non-zero weight to the string *The many cyclists*. However, their intersection computed with the Bar-Hillel construction is empty. To see this, note that all the paths from  $q_0$  to  $q_3$  contain the arc  $q_1 \xrightarrow{\varepsilon/0.3} q_2$ . Eq. (5d) will create a rule  $(q_1, \varepsilon, q_2) \xrightarrow{0.3} \varepsilon$ , but none of the rules produced by Eqs. (5b) and (5c) has the triplet  $(q_1, \varepsilon, q_2)$  on the right hand side. This misalignment results in an empty set of derivations in  $\mathfrak{G}_{\cap}$ . We illustrate more failure cases in App. A.

**Our generalized construction.** We now give an improved version of Nederhof and Satta's (2003) weighted intersection that handles  $\varepsilon$ -arcs in the WFSAs. For a WFSAs  $\mathcal{A}$  and a WCFG  $\mathfrak{G}$ , their intersection  $\mathfrak{G}_{\cap} = (\mathcal{N}_{\cap}, \Sigma, \mathcal{W}, S, \mathcal{P}_{\cap})$  is defined by the production rules in Construction 2 (in Fig. 3). Note that rules (7d) to (7f) are identical to rules (5b) to (5d) in the original construction. The initial rules in (7a) are a modified version of (5a), with the special symbol  $\tilde{S}$  instead of  $S$ ; this extra symbol handles the case of an  $\varepsilon$ -arc immediately before the final state. Rules (7b) form a left-branching structure that covers any such sequence of final consecutive  $\varepsilon$ -arcs. Rules (7c) transform a triplet with  $\tilde{S}$

<sup>7</sup>Note that this construction can handle multiple initial and final states, whereas Nederhof and Satta's (2003) construction assumes a WFSAs with a single initial and a single final state. A path's initial and final weights are taken into account by the weight of the rules produced by Eq. (5a).

### Construction 1.

$$S \xrightarrow{\lambda(q_I) \otimes \rho(q_F)} (q_I, S, q_F) \quad (5a)$$

$$\forall q_I \in I, \forall q_F \in F$$

$$(q_0, X, q_M) \xrightarrow{w} (q_0, X_1, q_1) \cdots (q_{M-1}, X_M, q_M)$$

$$\forall (X \xrightarrow{w} X_1 \cdots X_M) \in \mathcal{P} \quad (5b)$$

$$\forall q_0, \dots, q_M \in Q$$

$$(q_0, X, q_0) \xrightarrow{w} \varepsilon \quad (5c)$$

$$\forall (X \xrightarrow{w} \varepsilon) \in \mathcal{P}, \forall q_0 \in Q$$

$$(q_0, a, q_1) \xrightarrow{w} a \quad (5d)$$

$$\forall (q_0 \xrightarrow{a/w} q_1) \in \delta$$

Figure 2: The original Bar-Hillel construction.

into one with a standard  $S$ . Finally, rules (7g) form a right-branching structure that covers any other  $\varepsilon$ -arcs (not immediately before a final state).

As a final remark, we observe that—among the groups of rules listed in Fig. 3—the set of rules with maximum cardinality is the one defined by Eq. (7d). This set has cardinality  $\mathcal{O}(|\mathcal{P}||Q|^{M_*})$ , where  $M_*$  is 1 plus the length of the longest right-hand side among all the rules  $\mathcal{P}$ . All other equations in this construction lead to smaller sets of added rules, and hence, the asymptotic bound on the number of rules in the grammar remains unchanged compared to the original construction. In Fig. 1, we illustrate an example of how a path in the WFSa and a derivation in the WCFG relate to a derivation in the intersection. We now state the theorem of correctness.

**Definition 5.** Let  $\Sigma$  be an alphabet and  $\mathcal{W}$  be a commutative semiring. Let  $\mathfrak{G}$  be a WCFG and  $\mathcal{A}$  be a WFSa—both over  $\Sigma$  and  $\mathcal{W}$ . The **weighted join** of the derivations in  $\mathcal{D}_{\mathfrak{G}}$  with the paths in  $\mathcal{D}_{\mathcal{A}}$  is defined as:

$$(\mathcal{D}_{\mathfrak{G}} \bowtie \mathcal{D}_{\mathcal{A}}) \stackrel{\text{def}}{=} \left\{ \langle \mathbf{d}, \boldsymbol{\pi} \rangle \mid \mathbf{d} \in \mathcal{D}_{\mathfrak{G}}, \boldsymbol{\pi} \in \mathcal{D}_{\mathcal{A}} \quad (6) \right.$$

$$\left. \text{s.t. } \text{yield}(\mathbf{d}) = \text{yield}(\boldsymbol{\pi}) \right\}$$

with  $w(\langle \mathbf{d}, \boldsymbol{\pi} \rangle) = w(\mathbf{d}) \otimes w(\boldsymbol{\pi})$ .

**Theorem 1.** Let  $\mathfrak{G}$  be a WCFG and  $\mathcal{A}$  a WFSa over the same alphabet  $\Sigma$  and commutative semiring  $\mathcal{W}$ . Let  $\mathfrak{G}_{\cap}$  computed as in Construction 2. Then we have strong equivalence between  $\mathfrak{G}_{\cap}$  and  $\langle \mathfrak{G}, \mathcal{A} \rangle$ , i.e., there is a weight-preserving, yield-preserving bijection between  $\mathcal{D}_{\mathfrak{G}_{\cap}}$  and  $(\mathcal{D}_{\mathfrak{G}} \bowtie \mathcal{D}_{\mathcal{A}})$ .

*Proof.* See App. B for a detailed proof.  $\square$

### Construction 2.

$$S \xrightarrow{\lambda(q_I) \otimes \rho(q_F)} (q_I, \tilde{S}, q_F) \quad (7a)$$

$$\forall q_I \in I, \forall q_F \in F$$

$$(q_I, \tilde{S}, q_1) \xrightarrow{\mathbf{1}} (q_I, \tilde{S}, q_0)(q_0, \varepsilon, q_1) \quad (7b)$$

$$\forall q_I \in I, \forall q_0, q_1 \in Q$$

$$(q_I, \tilde{S}, q_0) \xrightarrow{\mathbf{1}} (q_I, S, q_0) \quad (7c)$$

$$\forall q_I \in I, \forall q_0 \in Q$$

$$(q_0, X, q_M) \xrightarrow{w} (q_0, X_1, q_1) \cdots (q_{M-1}, X_M, q_M)$$

$$\forall (X \xrightarrow{w} X_1 \cdots X_M) \in \mathcal{P} \quad (7d)$$

$$\forall q_0, \dots, q_M \in Q$$

$$(q_0, X, q_0) \xrightarrow{w} \varepsilon \quad (7e)$$

$$\forall (X \xrightarrow{w} \varepsilon) \in \mathcal{P}, \forall q_0 \in Q$$

$$(q_0, a, q_1) \xrightarrow{w} a \quad (7f)$$

$$\forall (q_0 \xrightarrow{a/w} q_1) \in \delta$$

$$(q_0, a, q_2) \xrightarrow{\mathbf{1}} (q_0, \varepsilon, q_1)(q_1, a, q_2) \quad (7g)$$

$$\forall a \in \Sigma, \forall q_0, q_1, q_2 \in Q$$

Figure 3: Our generalized Bar-Hillel construction.

**Corollary 1.**  $\mathfrak{G}_{\cap}$  and  $\langle \mathfrak{G}, \mathcal{A} \rangle$  are also weakly equivalent, meaning that:  $L_{\mathfrak{G}_{\cap}} = L_{\mathfrak{G}} \cap L_{\mathcal{A}}$ .

*Proof.* This follows trivially from Theorem 1.  $\square$

Theorem 1 may be seen as a generalization of Theorem 8.1 in Bar-Hillel et al. (1961) and Theorem 12 in Nederhof and Satta (2003). Indeed, Construction 1 is identical to Construction 2 when the WFSa is  $\varepsilon$ -free—modulo an unfold transformation (Tamaki and Sato, 1984) to reduce the resulting intersection grammar.

## 4 Conclusion

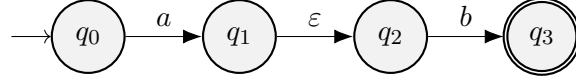
In this work, we discuss how the classic Bar-Hillel construction fails when the WFSa has  $\varepsilon$ -arcs. In this context, we propose a new generalized Bar-Hillel that handles these  $\varepsilon$ -arcs if they are present. Most importantly our construction is strongly equivalent to the product of the original WCFG and WFSa, i.e., for every derivation tree in the resulting grammar, it is always possible to uniquely associate it to a derivation tree in the input WCFG and to a path in the WFSa. Further, we give a full proof of correctness for our construction.

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## A Failure Cases of Original Construction

We distinguish two types of failure cases: (i)  $\text{supp}(L_{\mathfrak{G}_\cap}) \neq \text{supp}(L_{\mathcal{A}}) \cap \text{supp}(L_{\mathfrak{G}})$  and (ii)  $L_{\mathfrak{G}_\cap} \neq L_{\mathcal{A}} \cap L_{\mathfrak{G}}$ , both of which we will exemplify now. Notably, the case (ii) follows from (i), but—to be comprehensible—we will nonetheless give an example where (ii) fails without (i). For case (i), consider the following unweighted FSA:

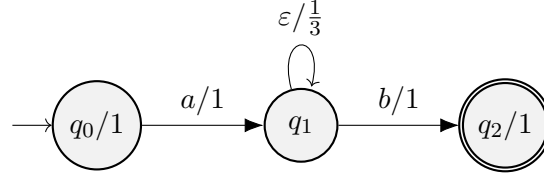


and the following unweighted CFG:

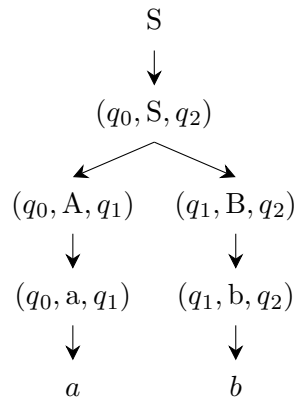
$$\begin{aligned} S &\rightarrow AB \\ A &\rightarrow a \\ B &\rightarrow b \end{aligned}$$

It is easy to see that the intersection of the language accepted by the FSA and the language generated by the CFG is  $\{ab\}$ . Construction 1, however, outputs an empty grammar (after useless rules removal) and, hence, an empty language. To see this, consider Eq. (5b) and Eq. (5d). First, Eq. (5d) will create a rule  $(q_1, \varepsilon, q_2) \rightarrow \varepsilon$ , but  $(q_1, \varepsilon, q_2)$  will be useless, as it cannot be reached from any of the rules produced by Eq. (5b). Second, Eq. (5b) will produce reachable non-terminals  $(q_0, A, q_i)$  and  $(q_i, B, q_3)$ , with  $i \in \{1, 2\}$ . The case of  $i = 1$  will reach  $a$  but not  $b$ , and  $i = 2$  will reach  $b$  but not  $a$ .

As stated above, to be comprehensive, we also show a case where only case (ii) fails, without (i). Take the following WFSA over the Inside semiring (Huang, 2008):



and the same grammar as above with weight 1 for all rules. It is easy to see that the language for  $s = ab$  in the WFSA is a geometric series  $L_{\mathcal{A}}(s) = \sum_{i=0}^{\infty} (\frac{1}{3})^i = \frac{3}{2}$ , while  $L_{\mathfrak{G}}(s) = 1$ . The output grammar  $\mathfrak{G}_\cap$  of Construction 1 will contain one single derivation  $d$ , as follows



with  $w(d) = 1$ , as all rules either stem from  $\mathfrak{G}$  or from the arcs  $q_0 \xrightarrow{a/1} q_1$  and  $q_1 \xrightarrow{b/1} q_2$ . There are no derivations rooted at  $S$  in  $\mathfrak{G}_\cap$  that correspond to the  $\varepsilon$ -arcs in  $\mathcal{A}$ : Similarly to the example above,  $(q_1, \varepsilon, q_1)$  will not be reachable. Hence, we have that  $L_{\mathcal{A}} \cap L_{\mathfrak{G}}(s) = L_{\mathcal{A}}(s) \otimes L_{\mathfrak{G}}(s) = \frac{3}{2}$ .

## B Proof of Theorem 1

Theorem 1 gives a result for derivations (which are always rooted at S) and paths (which always connect an initial state with a final state). However, in order to prove this theorem we must also consider subderivations and subpaths. We define subderivations as follows: a **subderivation**  $\tilde{d}$  is a sequence  $\alpha_0, \dots, \alpha_N$  with  $N \geq 0$ , where (i) in the case of  $N > 0$ ,  $\alpha_0 = X$ ,  $X \in \mathcal{N}$ , and  $\alpha_N \in \Sigma^*$ , such that for all  $0 < n \leq N$ , we have  $\alpha_{n-1} \xrightarrow{p_n}_L \alpha_n$  for some  $p_n \in \mathcal{P}$ , and (ii) in the case of  $N = 0$ ,  $\alpha_0 \in \Sigma \cup \{\varepsilon\}$ . The weight and yield of subderivations are defined analogously to that of derivations. In the extended case of  $N = 0$ , the yield is equal to  $\alpha_0$  and the weight is set to  $\mathbf{1}$ . We will say that a subderivation is **rooted** at X if  $\alpha_0 = X$ . We denote the set of subderivations rooted at X with  $\mathcal{D}_{\mathfrak{G}}(X)$ . Moreover, a subpath is defined as follows: A **subpath**  $\tilde{\pi}$  (of length  $N \geq 0$ ), is (i) in the case of  $N > 0$ , a sequence of arcs in  $\delta^*$  where the states of adjacent arcs are matched, and (ii) in the case of  $N = 0$  a single state  $q \in Q$ .<sup>8</sup> The subpath's weight, denoted  $\tilde{w}(\tilde{\pi})$ , is the product  $\tilde{w}(\tilde{\pi}) = \bigotimes_{n=1}^N w_n$  of the weights of the arcs along the subpath. In the extended case  $N = 0$  we set the weight to  $\mathbf{1}$  and the yield to  $\varepsilon$ . Note that, in contrast to the weight of a path, the weight of a subpath does not account for initial and final weights. The yield is defined identically to that of paths. We denote the set of all paths starting at  $q_i$  and ending at  $q_j$  with  $\mathcal{D}_{\mathcal{A}}(\{q_i, q_j\})$ . Note that the definitions of subderivation and subpath encapsulate the definitions of derivation and path respectively. Furthermore, we will denote with  $p(\pi)$  and  $n(\pi)$ , respectively, the first and the last state encountered along a path.

We will now prove two lemmas that will be necessary for the proof of Theorem 1.

**Lemma 1.** *For any triplet  $(q_0, X, q_m) \in \mathcal{N}_{\cap}$ , with  $X \neq \tilde{S}$  and  $q_0, q_m \in Q$ , there is a bijection  $\psi(\tilde{d}_{\cap}) = \langle \tilde{d}, \tilde{\pi} \rangle$  from  $\mathcal{D}_{\mathfrak{G}_{\cap}}((q_0, X, q_m))$  to the weighted join  $(\mathcal{D}_{\mathfrak{G}}(X) \bowtie \mathcal{D}_{\mathcal{A}}(\{q_0, q_m\}))$ , restricted to tuples in which the path does not have an  $\varepsilon$ -arc immediately before a final state. Moreover, it holds that:*

$$w(\tilde{d}_{\cap}) = w(\tilde{d}) \otimes \tilde{w}(\tilde{\pi}) \quad (8)$$

$$\text{yield}(\tilde{d}_{\cap}) = \text{yield}(\tilde{d}) = \text{yield}(\tilde{\pi}) \quad (9)$$

*Proof.* We begin by showing that  $\psi$  is well defined, that it is injective and that it satisfies the properties in Eqs. (8) and (9). We prove this by induction on subderivations.

*Lemma 1's Base Case.* We begin by observing that the only terminal rules from  $\mathcal{P}_{\cap}$  are defined by Eq. (7f) and Eq. (7e).

*Lemma 1's Base Case, Part #1.*  $\tilde{d}_{\cap}$  is obtained by the application of a single production rule  $(q_0, a, q_1) \xrightarrow{w} a$  from Eq. (7f). We define  $\psi(\tilde{d}_{\cap}) = \langle \tilde{d}, \tilde{\pi} \rangle$ , where  $\tilde{\pi} = q_0 \xrightarrow{a/w} q_1$  and  $\tilde{d} = a$  is the subderivation that contains just the string  $a$  with weight  $\mathbf{1}$ . It is easy to see that the yield is preserved. Moreover:

$$w(\tilde{d}_{\cap}) = w \quad (\text{by Eq. (7f)}) \quad (10a)$$

$$= w \otimes \mathbf{1} \quad (10b)$$

$$= \tilde{w}(\tilde{\pi}) \otimes w(\tilde{d}) \quad (10c)$$

*Lemma 1's Base Case, Part #2.*  $\tilde{d}_{\cap}$  is obtained by the application of a single production rule  $(q_0, X, q_0) \xrightarrow{w} \varepsilon$  from Eq. (7e). We construct  $\psi$  as follows:  $\psi(\tilde{d}_{\cap}) = \langle \tilde{d}, \tilde{\pi} \rangle$ , where  $\tilde{d} = X \xrightarrow{p}_L \varepsilon$  with  $p = X \xrightarrow{w} \varepsilon$ , and  $\tilde{\pi}$  is the subpath  $q_0$  with weight  $\mathbf{1}$ . Clearly the yield is preserved and:

$$w(\tilde{d}_{\cap}) = w \quad (\text{by Eq. (7e)}) \quad (11a)$$

$$= w \otimes \mathbf{1} \quad (11b)$$

$$= w(\tilde{d}) \otimes \tilde{w}(\tilde{\pi}) \quad (11c)$$

<sup>8</sup>We note the difference to paths defined in §2.2: a subpath does not need to start in an initial state and end in a final state.

*Lemma 1's Induction Step.* In the induction step, we show that the properties that we have shown for the base case propagate upwards along the derivation. In general, we will show that for any  $\tilde{\mathbf{d}}_\cap = (q_0, X, q_M) \xrightarrow{p}_L (q_0, X_1, q_1), \dots, (q_{M-1}, X_M, q_M) \Rightarrow_L \dots$ , we can construct  $\psi(\tilde{\mathbf{d}}_\cap) = \langle \tilde{\mathbf{d}}, \tilde{\pi} \rangle$  such that the mapping is injective and that properties Eqs. (8) and (9) hold. Additionally, as for the base case, we will show that  $\tilde{\pi}$  connects  $q_0$  with  $q_M$  and that  $\tilde{\mathbf{d}}$  is rooted at  $X$ . As our induction hypothesis, we will assume that each of these hypotheses hold for the subderivations rooted at each of the child non-terminals  $X_1, \dots, X_M$ .

*Lemma 1's Induction Step, Part #1.* The topmost rule applied in  $\tilde{\mathbf{d}}_\cap$  is  $p = (q_0, a, q_2) \xrightarrow{1} (q_0, \varepsilon, q_1)(q_1, a, q_2)$  defined by Eq. (7g). We denote with  $\tilde{\mathbf{d}}_{\cap,1}$  the subderivation rooted at  $(q_0, \varepsilon, q_1)$ , and we observe that the only possible form for this derivation is  $(q_0, \varepsilon, q_1) \xrightarrow{p} \varepsilon$  for some  $p = (q_0, \varepsilon, q_1) \xrightarrow{w} \varepsilon$ . We denote with  $\tilde{\mathbf{d}}_{\cap,2}$  the subderivation rooted at  $(q_1, a, q_2)$ , then by induction hypothesis, we know that there is a mapping  $\psi(\tilde{\mathbf{d}}_{\cap,2}) = \langle \tilde{\mathbf{d}}_2, \tilde{\pi}_2 \rangle$  such that Eqs. (8) and (9) are satisfied.

Then we construct  $\psi(\tilde{\mathbf{d}}_\cap) = \langle \tilde{\mathbf{d}}, \tilde{\pi} \rangle$ , so that  $\tilde{\mathbf{d}} = \tilde{\mathbf{d}}_2$  and  $\tilde{\pi} = q_0 \xrightarrow{\varepsilon/w} q_1 \circ \tilde{\pi}_2$ . As the yield of the subderivation rooted at  $(q_0, \varepsilon, q_1)$  is  $\varepsilon$ , the yield of  $\tilde{\mathbf{d}}_\cap$  is the same as that of  $\tilde{\mathbf{d}}_{\cap,2}$ . Further, the yield of  $\tilde{\pi}$  is the same as  $\tilde{\pi}_2$ . We thus have that:

$$\text{yield}(\tilde{\mathbf{d}}_\cap) = \text{yield}(\tilde{\mathbf{d}}_{\cap,2}), \quad \text{yield}(\tilde{\mathbf{d}}) = \text{yield}(\tilde{\mathbf{d}}_2), \quad \text{yield}(\tilde{\pi}) = \text{yield}(\tilde{\pi}_2) \quad (12)$$

By induction, we have that the yield is preserved. Similarly, we have that the weight is preserved:

$$w(\tilde{\mathbf{d}}_\cap) = \mathbf{1} \otimes w(\tilde{\mathbf{d}}_{\cap,1}) \otimes w(\tilde{\mathbf{d}}_{\cap,2}) \quad (13a)$$

$$= \mathbf{1} \otimes w \otimes w(\tilde{\mathbf{d}}_2) \otimes \tilde{w}(\tilde{\pi}_2) \quad (\text{induction hypothesis}) \quad (13b)$$

$$= w(\tilde{\mathbf{d}}_2) \otimes (w \otimes \tilde{w}(\tilde{\pi}_2)) \quad (\text{commutativity}) \quad (13c)$$

$$= w(\tilde{\mathbf{d}}) \otimes \tilde{w}(\tilde{\pi}) \quad (13d)$$

Finally by induction we assume that  $\tilde{\pi}_2$  connects state  $q_1$  with state  $q_2$ , which implies that  $\tilde{\pi}$  connects state  $q_0$  with state  $q_2$ .

*Lemma 1's Induction Step, Part # 2.* The topmost rule applied in  $\tilde{\mathbf{d}}_\cap$  is  $p = (q_0, X, q_M) \xrightarrow{w} (q_0, X_1, q_1), \dots, (q_{M-1}, X_M, q_M)$  defined by Eq. (7d). By induction we assume that the subderivation  $\tilde{\mathbf{d}}_{\cap,m}$  rooted at  $(q_{m-1}, X_m, q_m)$  is mapped by  $\psi$  into a derivation  $\tilde{\mathbf{d}}_m$  rooted at  $X_m$  and a path  $\tilde{\pi}_m$ , so that  $\text{yield}(\tilde{\mathbf{d}}_{\cap,m}) = \text{yield}(\tilde{\mathbf{d}}_m) = \text{yield}(\tilde{\pi}_m)$  and that  $w(\tilde{\mathbf{d}}_{\cap,m}) = w(\tilde{\mathbf{d}}_m) \otimes \tilde{w}(\tilde{\pi}_m)$ . We then define  $\psi(\tilde{\mathbf{d}}_\cap) = \langle \tilde{\mathbf{d}}, \tilde{\pi} \rangle$  where  $\tilde{\mathbf{d}} = X \xrightarrow{p}_L X_1, \dots, X_M \Rightarrow_L \dots$  with  $p = X \xrightarrow{w} X_1, \dots, X_M$  and  $\tilde{\pi} = \tilde{\pi}_1 \circ \dots \circ \tilde{\pi}_M$ . As the states of neighboring triplets are matched, and by induction we assume that  $\tilde{\pi}_m$  connects states  $q_{m-1}$  with state  $q_m$ , we have that  $\tilde{\pi}$  is a path from  $q_0$  to  $q_M$ . We note that the yield of  $\tilde{\mathbf{d}}$  is obtained by concatenation of  $\text{yield}(\tilde{\mathbf{d}}_m)$  from left to right, and that similarly the yield of  $\tilde{\pi}$  is obtained by concatenation of  $\text{yield}(\tilde{\pi}_m)$  from left to right. This, together with the induction hypothesis shows Eq. (9) of the lemma—as the yield of  $\tilde{\mathbf{d}}_\cap$  will also be given by the concatenation of  $\text{yield}(\tilde{\mathbf{d}}_{\cap,m})$  from left to right. We now show that Eq. (8) on weights hold:

$$w(\tilde{\mathbf{d}}_\cap) = w \otimes \bigotimes_{m=1}^M w(\tilde{\mathbf{d}}_{\cap,m}) \quad (14a)$$

$$= w \otimes \bigotimes_{m=1}^M w(\tilde{\mathbf{d}}_m) \otimes \tilde{w}(\tilde{\pi}_m) \quad (\text{induction hypothesis}) \quad (14b)$$

$$= \left( w \otimes \bigotimes_{m=1}^M w(\tilde{\mathbf{d}}_m) \right) \otimes \bigotimes_{m=1}^M \tilde{w}(\tilde{\pi}_m) \quad (\text{commutative property}) \quad (14c)$$

$$= w(\tilde{\mathbf{d}}) \otimes \tilde{w}(\tilde{\pi}) \quad (14d)$$



We have defined  $\psi$  in a bottom-up fashion. At each step changing the topmost rule would result either in a different tree  $\tilde{\mathbf{d}}$  or in a different path  $\tilde{\pi}$ , which proves injectivity. The proof that  $\psi$  is surjective is very similar, and consists in showing by induction, that for any  $\tilde{\mathbf{d}} \in \mathcal{D}_{\mathfrak{G}}(X)$ , and for any path  $\tilde{\pi}$  that does not have a sequence of  $\varepsilon$ -arc before a final state, it is always possible to build a derivation in  $\mathcal{D}_{\mathfrak{G}_\cap}((p(\pi), X, n(\pi)))$ . We limit ourselves to noting that it is always possible to do so by using rules from set Eqs. (7d) to (7f) as in the original Bar-Hillel construction, and by using rule defined by Eq. (7g) to cover  $\varepsilon$ -arcs in the WFSA.  $\square$

**Lemma 2.** *For any triplet  $(q_I, \tilde{S}, q) \in \mathcal{N}_\cap$ , with  $q_I \in I, q \in Q$ , there is a bijection  $\xi(\tilde{\mathbf{d}}_\cap) = \langle \tilde{\mathbf{d}}, \tilde{\pi} \rangle$  from  $\mathcal{D}_{\mathfrak{G}_\cap}((q_I, \tilde{S}, q))$  to the join  $(\mathcal{D}_{\mathfrak{G}}(S) \bowtie \mathcal{D}_{\mathcal{A}}(\{q_I, q\}))$ , and we have that:*

$$w(\tilde{\mathbf{d}}_\cap) = w(\tilde{\mathbf{d}}) \otimes \tilde{w}(\tilde{\pi}) \quad (15)$$

$$\text{yield}(\tilde{\mathbf{d}}_\cap) = \text{yield}(\tilde{\mathbf{d}}) = \text{yield}(\tilde{\pi}) \quad (16)$$

*Proof.* We now present an inductive proof (similar to the above) for this lemma.

*Lemma 2's Base Case.* The topmost rule applied in  $\tilde{\mathbf{d}}_\cap$  is  $(q_I, \tilde{S}, q) \xrightarrow{1} (q_I, S, q)$  from rules defined by Eq. (7c). We denote with  $\tilde{\mathbf{d}}_{\cap,1}$  the subderivation rooted at  $(q_I, S, q)$ . Then by Lemma 1, we know that there is a mapping  $\psi(\tilde{\mathbf{d}}_{\cap,1}) = \langle \tilde{\mathbf{d}}_1, \tilde{\pi}_1 \rangle$  such that Eqs. (15) and (16) are satisfied. We then define  $\xi(\tilde{\mathbf{d}}_\cap) = \langle \tilde{\mathbf{d}}_1, \tilde{\pi}_1 \rangle$ , and one can easily see that the properties in Eqs. (15) and (16) are satisfied.

*Lemma 2's Induction Step.* The topmost rule applied in  $\tilde{\mathbf{d}}_\cap$  is  $(q_I, \tilde{S}, q_1) \xrightarrow{1} (q_I, \tilde{S}, q_0)(q_0, \varepsilon, q_1)$  from rules defined by Eq. (7b). We denote with  $\tilde{\mathbf{d}}_{\cap,1}$  the subderivation rooted at  $(q_I, \tilde{S}, q_0)$ , and we assume by induction that  $\xi(\tilde{\mathbf{d}}_{\cap,1}) = \langle \tilde{\mathbf{d}}_1, \tilde{\pi}_1 \rangle$  and that properties in Eqs. (15) and (16) hold. We denote with  $\tilde{\mathbf{d}}_{\cap,2}$  the subderivation rooted at  $(q_0, \varepsilon, q_1)$ , and we observe that the only possible form for this derivation is  $(q_0, \varepsilon, q_1) \xrightarrow{p} \varepsilon$  for some  $p = (q_0, \varepsilon, q_1) \xrightarrow{w} \varepsilon$ . Then we can construct  $\xi(\tilde{\mathbf{d}}_\cap) = \langle \tilde{\mathbf{d}}, \tilde{\pi} \rangle$ , where  $\tilde{\mathbf{d}} = \tilde{\mathbf{d}}_1$  and  $\tilde{\pi} = \tilde{\pi}_1 \circ q_0 \xrightarrow{\varepsilon/w} q_1$ . The property in Eq. (16) is clearly satisfied, for property Eq. (15), we have:

$$w(\tilde{\mathbf{d}}_\cap) = \mathbf{1} \otimes w(\tilde{\mathbf{d}}_{\cap,1}) \otimes w(\tilde{\mathbf{d}}_{\cap,2}) \quad (17a)$$

$$= w(\tilde{\mathbf{d}}_{\cap,1}) \otimes w \quad (\text{weight of } \tilde{\mathbf{d}}_{\cap,2}) \quad (17b)$$

$$= w(\tilde{\mathbf{d}}_1) \otimes \tilde{w}(\tilde{\pi}_1) \otimes w \quad (\text{induction hypothesis}) \quad (17c)$$

$$= w(\tilde{\mathbf{d}}) \otimes \tilde{w}(\tilde{\pi}) \quad (\text{weight of } \tilde{\pi}) \quad (17d)$$

As for Lemma 1 we note that modifying the topmost rule in  $\tilde{\mathbf{d}}_\cap$ , would always result either in a different derivation  $\tilde{\mathbf{d}}$  or in a different path  $\tilde{\pi}$ , which proves injectivity. Surjectivity can be shown by induction, similarly to how we did for injectivity. We will simply note that given any derivation  $\tilde{\mathbf{d}}$  rooted at S, and given any path  $\tilde{\pi}$  starting from an initial state, it is always possible to build a matching derivation  $\tilde{\mathbf{d}}_\cap$  in  $\mathcal{D}_{\mathfrak{G}_\cap}((p(\pi), \tilde{S}, n(\pi)))$ , by using the result from Lemma 1, and applying rules defined by Eqs. (7b) and (7c).  $\square$

We can finally prove Theorem 1, which we restate here for convenience.

**Theorem 1.** *Let  $\mathfrak{G}$  be a WCFG and  $\mathcal{A}$  a WFSA over the same alphabet  $\Sigma$  and commutative semiring  $\mathcal{W}$ . Let  $\mathfrak{G}_\cap$  computed as in Construction 2. Then we have strong equivalence between  $\mathfrak{G}_\cap$  and  $\langle \mathfrak{G}, \mathcal{A} \rangle$ , i.e., there is a weight-preserving, yield-preserving bijection between  $\mathcal{D}_{\mathfrak{G}_\cap}$  and  $(\mathcal{D}_{\mathfrak{G}} \bowtie \mathcal{D}_{\mathcal{A}})$ .*

*Proof.* Any derivation  $\tilde{\mathbf{d}}_\cap$  in  $\mathcal{D}_{\mathfrak{G}_\cap}(S)$  takes the form  $S \xrightarrow{p} (q_I, \tilde{S}, q_F) \Rightarrow_L \dots$  with  $p = S \xrightarrow{\lambda(q_I) \otimes \rho(q_F)} (q_I, \tilde{S}, q_F)$ , for  $q_I \in I$  and  $q_F \in F$ . We denote with  $\tilde{\mathbf{d}}_\cap$  the subderivation rooted at  $(q_I, \tilde{S}, q_F)$ . We can thus define  $\phi(\tilde{\mathbf{d}}_\cap) = \langle \tilde{\mathbf{d}}, \tilde{\pi} \rangle = \langle \tilde{\mathbf{d}}, \tilde{\pi} \rangle$ , where  $\xi(\tilde{\mathbf{d}}_\cap) = \langle \tilde{\mathbf{d}}, \tilde{\pi} \rangle$ , and  $\xi$  is the bijection defined in Lemma 2. By Lemma 2 we have that  $\tilde{\mathbf{d}} = \mathbf{d}$  is rooted at S, and that  $\tilde{\pi} = \pi$  has

initial and final states:  $p(\tilde{\pi}) = q_I$  and  $n(\tilde{\pi}) = q_F$ . Clearly,  $\text{yield}(\mathbf{d}_\cap) = \text{yield}(\tilde{\mathbf{d}}_\cap)$  and by Lemma 2  $\text{yield}(\tilde{\mathbf{d}}_\cap) = \text{yield}(\tilde{\mathbf{d}}) = \text{yield}(\tilde{\pi})$ . Further, by definition  $\text{yield}(\mathbf{d}) = \text{yield}(\tilde{\mathbf{d}})$  and  $\text{yield}(\tilde{\pi}) = \text{yield}(\pi)$ . Moreover, we have that:

$$w(\mathbf{d}_\cap) = w(p) \otimes w(\tilde{\mathbf{d}}_\cap) \quad (\text{weight of a derivation}) \quad (18a)$$

$$= w(p) \otimes w(\tilde{\mathbf{d}}) \otimes \tilde{w}(\tilde{\pi}) \quad (\text{Lemma 2}) \quad (18b)$$

$$= \lambda(q_I) \otimes \rho(q_F) \otimes w(\tilde{\mathbf{d}}) \otimes \tilde{w}(\tilde{\pi}) \quad (\text{weight of } p) \quad (18c)$$

$$= w(\tilde{\mathbf{d}}) \otimes \lambda(q_I) \otimes \tilde{w}(\tilde{\pi}) \otimes \rho(q_F) \quad (\text{commutative property}) \quad (18d)$$

$$= w(\mathbf{d}) \otimes w(\pi) \quad (\text{definition of weight of a path}) \quad (18e)$$

which proves that  $\phi$  is weight and yield preserving. By Lemma 2 we know that  $\xi$  is a bijection, which implies that modifying the topmost rule  $p$  would result in a different tuple  $\langle \mathbf{d}, \pi \rangle$ . This proves the injectivity of  $\phi$ . Conversely, consider any path  $\pi$  connecting an initial state with a final one and any derivation  $\mathbf{d}$  rooted at  $S$ , such that  $\text{yield}(\mathbf{d}) = \text{yield}(\pi)$ . By Lemma 2 we know that it is always possible to construct a subderivation  $\tilde{\mathbf{d}}_\cap$ , rooted at  $(q_I, \tilde{S}, q_F)$ , that satisfies Eqs. (15) and (16). Thus we can construct  $\mathbf{d}_\cap = S \xrightarrow{p} (q_I, \tilde{S}, q_F) \Rightarrow_L \cdots$  with  $p = S \xrightarrow{\lambda(q_I) \otimes \rho(q_F)} (q_I, \tilde{S}, q_F)$  a rule from Eq. (7a). This shows the surjectivity of  $\phi$ . □